Review, Problem 10

Let a_n = nln(1+1/n). Given any ε > 0 find a N(ε) such that

$|a_n - 1| < \varepsilon$ for all n >= N(ε).

Solution: We start with the Taylor series for 0< x < 1:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

Now if 0 < x < 1 then

$$\frac{x^n}{n} - \frac{x^{n+1}}{n+1} = x^n\left(\frac{1}{n} - \frac{x}{n+1}\right) > 0$$

Consequently

$$\ln(1+x) = x - \left(\frac{x^2}{2} - \frac{x^3}{3}\right) - \left(\frac{x^4}{4} - \frac{x^5}{5}\right) - \ldots < x$$

or

x -ln(1+x) >0.

Similarly,

$$\ln(1+x) = x - \frac{x^2}{2} + \left(\frac{x^3}{3} - \frac{x^4}{4}\right) + \left(\frac{x^5}{5} - \frac{x^6}{6}\right) + \;\; > x - \frac{x^2}{2}$$

So

$x - \ln(1+x) < x^2/2$.

To summarize:

$0 < x - \ln(1+x) < x^2/2$.

If we set x = 1/n here we have

$$0 < \frac{1}{n} - \ln\left(1 + \frac{1}{n}\right) < \frac{1}{2n^2}$$

$$\Rightarrow$$

$$0 < 1 - n\ln\left(1 + \frac{1}{n}\right) < \frac{1}{2n}$$

This implies that nln(1+1/n) -> 1.

Furthermore

$$\frac{1}{2n} < \varepsilon \Leftrightarrow n > \frac{1}{2\varepsilon}$$

so we take N(ε) = 1 +[1/(2ε)]. Then

$|a_n - 1| < \varepsilon$ for all n >= N(ε)