Review, Problem 1

Let A = B = the set of positive integers and A x B be the ir cartesian product. Write an algorithm for listing the elements of A x B as a sequence, A x B = {c_n : n = 1, 2, ..}, where c_n is the n^{th} element of A x B.

Using your algorithm, determine the 52^{nd} element of A x B.

Solution: Since

A x B = {(a,b): a, b = positive integers}

we can organize the couples (a, b) according to the sums a+b:

	sum	no. of c_n
c_1 = (1,1)	2	1
c_2 = (1,2) c_3 = (2,1)	3	2
c_4 = (1,3) c_5 = (2,2) c_6 = (3,1)	4	3
. .		
(1,n-1)		
(2,n-2)		
	n	
(k,n-k)		
(n-1,1)		n-1

Since the total number of (a,b) with a = b <= n is

$$1+2+3+..+(n-1) = (n-1)n/2 = S(n)$$

it follows that the last element listed above will be indexed by

$$C_{n(n-1)/2} = (n-1, 1).$$

Tabulating the last element in each block, $C_{n(n-1)/2}$, we have

n	n(n-1)/2	$C_{n(n-1)/2}$
2	1	c_1 = (1,1)
3	3	c_3 = (2,1)
4	6	c_6 = (3,1)
5	10	c_{10} = (4,1)
6	15	c_{15} = (5,1)
7	21	c_{21} = (6,1)
8	28	c_{28} = (7,1)
9	36	c_{36} = (8,1)
10	45	c_{45} = (9,1)
11	55	c_{55} = (10,1)

From this it follows that the 52nd element of A x B will be an (a, b) with a + b = 11. Counting backward:

$$c_{55} = (10,1),\ c_{54} = (9,2),\ c_{53} = (8,3),\ c_{52} = (7,4).$$

Thus the 52nd element is (7,4).

As for the general indexing problem, it is given by:

Set S(n) = n(n-1)/2, then the elements having a + b = n + 1 will be indexed by

	sum	no. of c_n
$C_{S(n)+1} = (1,n)$		
$C_{S(n)+2} = (2,n-1)$		
.	n+1	
$C_{S(n)+k} = (k,n+1-k)$		
$C_{S(n)+n} = (n,1)$		n