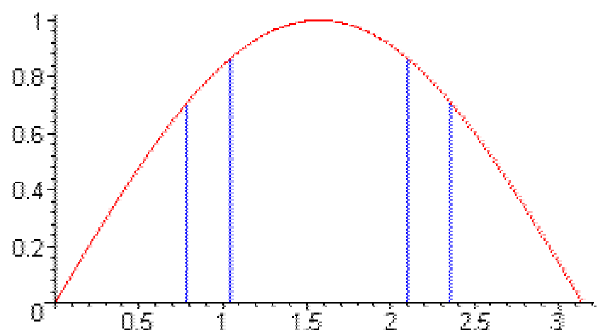
Page 86, Problem 6

Let $f(x)$ be a continuous function on a finite interval $[a, b]$. Suppose that $f(a) < f(b)$ and choose c and d so that $f(a) < c < d < f(b)$. Let

$$S = \{x : c \leq f(x) \leq d\}.$$

(a) Show by example that S need not be a single interval.
(b) Suppose that $f(x)$ is increasing on $[a, b]$. Prove that in this case S is a single interval.

Proof: (a) Consider the function $f(x) = \sin(x)$ on $[0, Pi]$:



Take $c = \sin(Pi/4) = 1/sqrt(2)$ and $d = \sin(Pi/3) = sqrt(3)/2$. Then

$$S = \{ x: 1/\sqrt{2} \leq \sin(x) \leq \sqrt{3}/2\} = \{x : \pi/4 \leq x \leq \pi/3\}\cup\{x : 3\pi/4 \leq x \leq 2\pi/3\}$$

is the union of two disjoint intervals.

(b) Suppose $f(x)$ is increasing on $[a, b]$ and

$$S = \{x : c \leq f(x) \leq d\}.$$

Then, since $c < d$, the intermediate value theorem guarantees that S is not empty. So let

$$x_0 = \inf(S) \qquad x_1 = \sup(S).$$

Now, $f(x_0) \geq c$:

Let $m = 1, 2, ..$ take $x_0 \leq x_m \leq x_0 + 1/m$ and x_m in S.
Then, since f is increasing, $f(x_0) \leq f(x_m)$.
Since x_m is in S. $f(x_m) \geq c$.
Since $x_m \to x_0$ and $f(x)$ is continuous,

it follows that $f(x_0) \geq c$

Similarly $f(x_0) \leq d$.

To continue, let x be any number in $[x_0, x_1]$. Then

$$
\begin{aligned}
&x_0 \le x \le x_1 \Rightarrow \\
&c \le f(x_0) \le f(x) \le f(x_1) \le d \Rightarrow \\
&x \text{ is in } S \Rightarrow \\
&[x_0, x_1] \subset S
\end{aligned}
$$

Finally if $x < x_0$ then x is not in S so $f(x) < c$ and if $x > x_1$ then $f(x) > d$. Consequently, $S \subset [x_0, x_1]$.

Thus $S = [x_0, x_1]$.