Page 79, Problem 8

Let $f(x) = x^2$ and let $\varepsilon > 0$ be given.

a) Find a $\delta > 0$ such that $|x-1| < \varepsilon \Rightarrow |f(x) - f(1)| < \varepsilon$.
b) Find a $\delta > 0$ such that $|x-2| < \varepsilon \Rightarrow |f(x) - f(2)| < \varepsilon$.

Proof: (a) First

$$| f(x) - f(1) | = | x^2 - 1^2 | = | (x-1)(x+1) | .$$

Next, since we are working with x in the neighborhood of 1 we can assume that $0 \le x \le 2$; that is, $|x+1| \le 3$. Consequently

$$| f(x) - f(1) | = | (x-1)(x+1) | \le 3 | x-1 | .$$

Finally

$$3 | x-1 | < \varepsilon \Leftrightarrow | x-1 | < \varepsilon / 3 ,$$

so take $\delta = \varepsilon / 3$.

b)

$$| f(x) - f(2) | = | x^2 - 2^2 | = | (x-2)(x+2) |$$

Next, since we are working with x in the neighborhood of 2 we can assume that $1 \le x \le 3$; that is, $|x+2| \le 5$. Consequently

$$| f(x) - f(2) | = | (x-2)(x+2) | \le 5 | x-2 | .$$

Finally

$$5 | x-2 | < \varepsilon \Leftrightarrow | x-5 | < \varepsilon / 5 ,$$

so take $\delta = \varepsilon / 5$.