Page 59, Problem 6

Consider the sequence $a_1 = 1/2$, $a_2 = 1/4$, $a_3 = 1/2$, $a_4 = 3/4$, the next seven terms are 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8... and so forth. What are the limit points of the sequence.

Solution: All the numbers in the closed interval [0, 1] are limit points of this sequence.

Note first: the sequence contains all numbers of the form

$$\frac{1}{2^2}, \frac{2}{2^2}, \frac{3}{2^2}, \ldots \frac{k}{2^2}, \frac{k+1}{2^2}, \ldots \frac{2^n-1}{2^n}$$

Second, the difference between consecutive members of the sequence displayed above is $1/2^n$. Thus if x is any number in [0, 1] and n is chosen so that $1/2^n < \varepsilon$ then the interval

$$(x - \varepsilon, x + \varepsilon)$$

contains at least one number of the form $k/2^n$. Thus any x in [0, 1] is a limit point of the sequence.