Page 51, Problem 13

Let $\{a_n\}$ be a monotone, bounded sequence and define b_n by

$$b_n = \frac{a_1 + a_2 + \ \ldots. + a_n}{n}.$$

Prove that $\{b_n\}$ is a monotone, bounded sequence.

Proof: Asssume that $\{a_n\}$ is an increasing sequence, so

$$a_1 \le a_2 \le \ \ldots \le a_n \le a_{n+1} \le \ldots$$

(A similar proof holds when $\{a_n\}$ is decreasing)

Working backward

$$\frac{a_1 + a_2 + \ \ldots. + a_n}{n} \le \frac{a_1 + a_2 + \ \ldots. + a_n + a_{n+1}}{n+1}$$

$$\Leftrightarrow$$

$$\left(\frac{a_1 + a_2 + \ \ldots. + a_n}{1}\right)\left(\frac{1}{n} - \frac{1}{n+1}\right) \le \frac{a_{n+1}}{n+1}$$

$$\Leftrightarrow$$

$$\left(\frac{a_1 + a_2 + \ \ldots. + a_n}{1}\right)\left(\frac{1}{n(n+1)}\right) \le \frac{a_{n+1}}{n+1}$$

$$\Leftrightarrow$$

$$a_1 + a_2 + \ \ldots. + a_n \le n a_{n+1}$$

This last inequality is true for

$$\begin{array}{ll} a_{n+1} \ge a_1 & \\ a_{n+1} \ge a_2 & \\ . & \Rightarrow n a_{n+1} \ge a_1 + a_2 + \ \ldots. + a_n \\ . & \\ a_{n+1} \ge a_n & \end{array}$$