page 51, Problem 10\

Let a_1 = sqrt(2) and

$$a_{n+1} = \sqrt{2+\sqrt{a_n}}\ .$$

(a) Prove that sqrt(2) ≤ a_n ≤ 2 for all n.
(b) Prove that $\{a_n\}$ is a Cauchy sequence.

Proof: (a) By the induction hypothesis

$$a_{n+1} = \sqrt{2+\sqrt{a_n}} \geq \sqrt{2}$$
$$a_{n+1} = \sqrt{2+\sqrt{a_n}} \leq \sqrt{2+\sqrt{2}} \leq \sqrt{3.5} \leq 2 \ .$$

(b)Some algebraic manipulation is required: First

$$a_{n+1}{}^2 - a_n{}^2 = (2+\sqrt{a_n}) - (2+\sqrt{a_{n-1}})$$
$$= (\sqrt{a_n} - \sqrt{a_{n-1}})\frac{\sqrt{a_n}+\sqrt{a_{n-1}}}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$
$$= \frac{a_n - a_{n-1}}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

So

$$a_{n+1} - a_n = \frac{a_n - a_{n-1}}{(\sqrt{a_n}+\sqrt{a_{n-1}})(a_{n+1}+a_n)}$$

By part (a)

$$(\sqrt{a_n}+\sqrt{a_{n-1}})(a_{n+1}+a_n) \geq$$

$$(\sqrt{\sqrt{2}}+\sqrt{\sqrt{2}})(\sqrt{2}+\sqrt{2}) = 2\sqrt{\sqrt{2}}\,2\sqrt{2} \geq 4$$

Consequently

$$|a_{n+1} - a_n| = \frac{|a_n - a_{n-1}|}{(\sqrt{a_n}+\sqrt{a_{n-1}})(a_{n+1}+a_n)} \leq \frac{|a_n - a_{n-1}|}{4}$$

Iterating this inequality:

$$|a_3 - a_2| \le \frac{|a_2 - a_1|}{4}$$

$$|a_4 - a_3| \le \frac{|a_3 - a_2|}{4} \le \frac{|a_2 - a_1|}{4^2}$$

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$$|a_n - a_{n-1}| \le \frac{|a_2 - a_1|}{4^{n-2}}$$

Then, as in problem 7,

$$\begin{aligned} |a_{n+1} - a_{n-1}| &= |a_{n+1} - a_n + a_n - a_{n-1}| \\ &\le |a_{n+1} - a_n| + |a_n - a_{n-1}| \\ &\le \frac{1}{4^{n-1}} + \frac{1}{4^{n-2}} \end{aligned}$$

and, more generally,

$$\begin{aligned} |a_{n+k} - a_{n-1}| &\le \frac{1}{4^{n+k-2}} + \frac{1}{4^{n+k-3}} + \quad + \frac{1}{4^{n-2}} \\ &< \frac{1}{4^{n-2}}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \ \dots\ \right) \\ &= \frac{1}{4^{n-2}}\frac{4}{3} \end{aligned}$$

That is, replacing n+k by m and n-1 by n, for m > n

$$|a_m - a_n| < \frac{3}{4^{n-2}}\ .$$

Since n is arbitrary, it follows that $\{a_m\}$ is a Cauchy sequence.