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(a) Let

$$a_{n+1} = \frac{a_n}{2} + 2, \quad a_1 = \frac{1}{2}$$

Prove that $a_n \to 4$.

Proof:

$$a_{n+1} - 4 = \frac{a_n}{2} - 2 = \frac{a_n - 4}{2} \Rightarrow$$

$$a_{n+1} - 4 = \frac{a_n - 4}{2} \quad for \ \ n = 1, 2, 3, \ldots$$

Iterating this last equation we have:

$$a_2 - 4 = \frac{1}{2}(a_1 - 4) \Rightarrow$$

$$a_3 - 4 = \frac{1}{2}(a_2 - 4) = \frac{1}{2^2}(a_1 - 4) \Rightarrow$$

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$$a_n - 4 = \frac{1}{2^{n-1}}(a_1 - 4)$$

Consequently

$$a_n - 4 = \frac{1}{2^{n-1}}(a_1 - 4) \to 0$$

(b) Suppose that

$$a_{n+1} = \lambda a_n + 2$$

where $|\lambda| < 1$ and a_1 is any fixed number. Prove that

$$a_n \to 2/(1 - \lambda)$$

Proof: Iterate:

$$a_2 = \lambda a_1 + 2$$

$$a_3 = \lambda a_2 + 2 = \lambda^2 a_1 + 2\lambda + 2$$

$$a_4 = \lambda a_3 + 2 = \lambda^3 a_1 + 2\lambda^2 + 2\lambda + 2$$

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$$a_n = \lambda^{n-1} a_1 + 2(\lambda^{n-1} + \lambda^{n-2} + \quad + \lambda + 1)$$

$$= \lambda^{n-1} a_1 + 2\frac{1 - \lambda^n}{1 - \lambda} \to \frac{2}{1 - \lambda}$$

The limiting values follow from the assumption that $|\lambda| < 1$.