Page 25, Problem 8

Prove that for all positive integers n

$$1^3+2^3+\ \dots\ +n^3=(1+2+\dots.+n)^2$$

The proof is a bit tricky in that you need to recall that

$$1+2+\ \dots+n=n(n+1)/2$$

So, the problem is equivalent to proving that

$$1^3+2^3+\ \dots\ +n^3=(n(n+1)/2)^2\,,$$

which is true for n = 1.

As for the inductive step: First

$$1^3+2^3+\ \dots\ +n^3=(n(n+1)/2)^2$$
$$\Rightarrow$$
$$1^3+2^3+\ \dots\ +n^3+(n+1)^3=\left(n\frac{n+1}{2}\right)^2+(n+1)^3$$

Second,

$$\left(n\frac{n+1}{2}\right)^2+(n+1)^3$$
$$=$$
$$(n+1)^2\left(\frac{n^2}{4}+n+1\right)$$
$$=$$
$$(n+1)^2\frac{n^2+4n+4}{4}$$
$$=$$
$$(n+1)^2\frac{(n+2)^2}{4}=\left(\frac{(n+1)(n+2)}{2}\right)^2$$

Thus

$$1^3+2^3+\ \dots\ +n^3+(n+1)^3=\left(\frac{(n+1)(n+2)}{2}\right)^2$$

which completes the proof of the inductive step.