

Suppose that $0 < a < b$. Prove that:

$$(a) \ a < \sqrt{ab} < b$$

$$(b) \ \sqrt{ab} < \frac{a+b}{2}$$

Proof of (a):

$$b > a > 0 \Rightarrow b^2 > ba \Rightarrow b > \sqrt{ba}$$

$$0 < a < b \Rightarrow a^2 < ab \Rightarrow a < \sqrt{ab}$$

Proof of (b):

$$(\sqrt{b} - \sqrt{a})^2 > 0$$

$$\Rightarrow$$

$$(\sqrt{b} - \sqrt{a})^2 = b - 2\sqrt{a}\sqrt{b} + a > 0$$

$$\Rightarrow$$

$$b + a > 2\sqrt{ab}$$

$$\Rightarrow$$

$$\sqrt{ab} < \frac{a+b}{2}$$