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The basic problem is to show that the set of functions from the integers to the set {1,2} is uncountable.

Solution: To define f from the integers to {1, 2}

$$f: \{1, 2,, .., n, ..\} \to \{1,2\}$$

you must first decide what set of integers will be mapped onto 1. Then the complement will be mapped onto 2.

So, defining the function is equivalent to selecting a (possibly infinite) subset of {1, 2,, .., n, ..}. Suppose we select a subset A, letting $a_n = 1$ if n is in A and $a_n = 0$ if n is not in A. Thus, selecting a subset of {1, 2,, .., n, ..} is equivalent to selecting a sequence of 0s and 1s.

But the set of all sequences $a = \{a_1, a_2, .., a_n, ..\}$ where each a_n is a 0 or 1 is uncountable:

Suppose the set were countable. Then we could write

$$A = \{b_n, n = 1,2, ..\}$$

where each

$$b_n = \{b_{n1}, b_{n2}, .., b_{nn}, ..\}$$

is a sequence of 0s and 1's. In addition every sequence of 0s and 1s is a b_m for some m.

Construct a new sequence $c = \{c_1, c_2, .., c_n , ..\}$ 0s and 1s by setting

$$c_n = 1 \text{ if if } b_{nn} = 0$$
$$c_n = 0 \text{ if } b_{nn} = 1$$

Then c is not equal to b_n for any n, a contradiction.

Defining f from the integers to any finite set is the same problem because you must first define set of integers that map onto 1 and, by the above argument, this can be done in an uncountable number of ways