Let

$$Q = \{ f(x) = x^2 + ax + b : a = 1,2,3 ..; b = 1,2,3\}$$

and

$$R = \{r : f(r) = 0 \text{ for some } f(x) \text{ in } Q\}.$$

a) Prove that R is countable.
b) Assuming that $R = \{r_n : n = 1, 2, 3, ..\}$ describe an alogorithm for determining $r_n$, the $n^{th}$ element of R. (You need only give the general idea of your algorithm; don't get hung up on details.
c) Using this algorithm write down $r_1,...,r_6$, the first 6 elements of R.

Solution: (a) Map

$$x^2 + ax + b \to (a,b).$$

This is a 1-1 map of Q onto I x I, the cartesian product of the positive integers, a set known to be countable. Thus Q is countable.

So, let

$$Q = \{q_n(x) = x^2 + a_n x + b_n \}.$$

Then let

$$R_n = \{r: r^2 + a_n r + b_n = 0\}.$$

Then $R_n$ is a set of 1 or 2 elements, depending on whether the underlying polynomial has 1 or 2 distinct roots.

Finally $R = \bigcup_{n=1}^{\infty} R_n$.

That is, R is the union of a countable number of finite sets, and is itself an infinite set, so R is countable.

(b) The organizing principle would be to first arrange the polynomials $x^2 + ax + b$ by the sum of the coefficients, find the roots, and then index the roots.

(c) Start with $x^2 + x + 1 = 0$. There are two roots:

$$x = \frac{-1 \pm i\sqrt{3}}{2}; \quad \text{set } r_1 = \frac{-1 + i\sqrt{3}}{2}, \quad r_2 = \frac{-1 - i\sqrt{3}}{2}$$

Consider next $x^2 + x + 2 = 0$. There are two roots:

$$x = \frac{-1 \pm i\sqrt{7}}{2}; \quad \text{set } r_3 = \frac{-1 + i\sqrt{7}}{2}, \quad r_4 = \frac{-1 - i\sqrt{7}}{2}$$

Next $x^2 + 2x + 1 = (x + 1)^2 = 0$ has one root:

$$x = -1; \quad \text{set } r_5 = -1.$$

Finally $x^2 + 2x + 2 = 0$ has two roots:

$$x = -1 \pm i; \quad \text{set } r_6 = -1 - i$$