8. Convergence in probability

170B Probability Theory, Puck Rombach

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Bertsekas & Tsitsiklis: Section 5.3. Assumed knowledge: Some analysis.

We have seen in the previous topics that, if we take a sample of size \( n \), the sequence of sample means \( M_1, M_2, M_3, M_4, \ldots \) seems to “converge” in some sense to \( \mu \), but what do we mean by that exactly?

We recall the definition of convergence for deterministic sequences.

Let \( \{a_n\} \) be a sequence of real numbers, we say that

- \( \{a_n\} \text{ converges to } a \), or
- \( \lim_{n \to \infty} a_n = a \), or
- \( a_n \to a \text{ as } n \to \infty \),

if

\[ \forall \epsilon > 0, \exists n_0 \text{, such that } |a_n - a| \leq \epsilon \forall n \geq n_0. \]

Similarly, we now define probabilistic convergence in exactly the same way as what we saw in the weak law of large numbers.

Let \( \{X_n\} \) be a sequence of random variables. We say that

- \( \{X_n\} \text{ converges to } a \text{ in probability, or} \)
- \( \text{p-lim}_{n \to \infty} X_n = a \), or
- \( X_n \xrightarrow{p} a \text{ as } n \to \infty \),

if

\[ \forall \epsilon, \delta > 0, \exists n_0 \text{, such that } \mathbb{P}(|X_n - a| \geq \epsilon) \leq \delta \forall n \geq n_0, \]

or, equivalently

\[ \forall \epsilon > 0, \lim_{n \to \infty} \mathbb{P}(|X_n - a| \geq \epsilon) = 0. \]

Notice that the second definition simply asks for deterministic convergence of \( \mathbb{P}(|X_n - a| \geq \epsilon) \).
Exercise 8.1. See example 5.8 from the book. Let $X_n$ be a random variable with

$$p_{X_n}(x) = \begin{cases} \frac{1}{n} & \text{if } x = 1, \\ 1 - \frac{1}{n} & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Does $X_n$ converge in probability to a constant? Does $\mathbb{E}(X_n)$ converge deterministically to the a constant? If so, is it the same constant?

Now, answer the same questions for the sequence of random variables $X_n$ with

$$p_{X_n}(x) = \begin{cases} \frac{1}{n} & \text{if } x = n, \\ 1 - \frac{1}{n} & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and then for the sequence of random variables $X_n$ with

$$p_{X_n}(x) = \begin{cases} \frac{1}{n} & \text{if } x = n^2, \\ 1 - \frac{1}{n} & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 8.2. (See also example 5.6 in the book.) Let $X_1, X_2, X_3, \ldots$ be iid random variables, which are all continuous and uniform on $[0, 1]$. Let $Y_n = \max(X_1, X_2, \ldots, X_n)$. Does $Y_n$ converge in probability?

Exercise 8.3. Suppose $X_n \xrightarrow{p} a$ and $Y_n = \max(X_n, b)$. Does $Y_n$ converge in probability to a constant?

Exercise 8.4. Suppose $X_n \xrightarrow{p} a$ and $Y_n \xrightarrow{p} b$. Does the random variable $Z_n = X_n + Y_n$ converge in probability to a constant?

We show some of the following general properties of convergence in probability in the lecture and in the exercises.

Lemma 8.5. If $\{X_n\}$ and $\{Y_n\}$ are such that $X_n \xrightarrow{p} a$ and $Y_n \xrightarrow{p} b$, where $a, b \in \mathbb{R}$, then

- $cX_n + dY_n \xrightarrow{p} ca + db$,
- $|X_n| \xrightarrow{p} |a|$,
- $X_nY_n \xrightarrow{p} ab$,
- $\max(X_n, Y_n, c) \xrightarrow{p} \max(a, b, c)$.

Recommended exercises

- Example 5.6-5.8 (p.272)
- Problem 5-6 (p.288-289)
Solutions to Exercises

Solution 8.1. In all three cases, we have that $X_n \xrightarrow{p} 0$.

In the first case, $E(X_n) \rightarrow 0$.

In the second case, $E(X_n) \rightarrow 0$ (in fact, it’s always equal to 1). It is surprising that $E(X_n)$ converges to a constant deterministically, but a constant that is different from the constant that $X_n$ converges to in probability.

In the third case, we have $E(X_n) = n$, which does not converge at all.

We learn from this example that convergence in probability of a random variable does not guarantee any behavior of the first moment of that random variable.

Solution 8.2. We would expect that $Y_n \xrightarrow{p} 1$, and we check that, indeed, for any $\epsilon > 0$,

$$
P(|Y_n - 1| \geq \epsilon) = P(Y_n \leq 1 - \epsilon)
= P(X_1 \leq 1 - \epsilon, X_2 \leq 1 - \epsilon, \ldots, X_n \leq 1 - \epsilon)
= P(X_1 \leq 1 - \epsilon) \cdot P(X_2 \leq 1 - \epsilon) \cdot \ldots \cdot P(X_n \leq 1 - \epsilon)
= (1 - \epsilon)^n \rightarrow 0.
$$

Solution 8.3. We show that $Y_n \xrightarrow{p} \max(a, b)$, which is what we would expect intuitively.

Suppose that $a \leq b$. Then,

$$
P(|Y_n - b| \geq \epsilon) = P(Y_n \geq b + \epsilon)
= P(X_n \geq b + \epsilon)
= P(X_n \geq a + b - a + \epsilon)
= P(X_n - a \geq b - a + \epsilon)
\leq P(|X_n - a| \geq b - a + \epsilon)
\leq P(|X_n - a| \geq \epsilon') \rightarrow 0,
$$

because $\epsilon' = b - a + \epsilon > 0$.

Suppose that $a > b$. Then,

$$
P(|Y_n - a| \geq \epsilon) \leq P(|X_n - a| \geq \epsilon) \rightarrow 0.
$$
Solution 8.4 We would expect that $Z_n \xrightarrow{p} a + b$. We have, for any $\epsilon > 0$,

$$
P(|Z_n - (a + b)| \geq \epsilon) = P(|X_n - a + Y_n - b| \geq \epsilon).
$$

Now, we have the following event relation:

$$
|X_n - a + Y_n - b| \geq \epsilon \subseteq |X_n - a| \geq \epsilon/2 \cup |Y_n - b| \geq \epsilon/2.
$$

Check carefully that you agree with this statement. The following also works fine:

$$
|X_n - a + Y_n - b| \geq \epsilon \subseteq |X_n - a| \geq \epsilon/4 \cup |Y_n - b| \geq 3\epsilon/4.
$$

Also, remember that, if $A \subseteq B$, then $P(A) \leq P(B)$. Also, $P(A \cup C) \leq P(A) + P(C)$. This gives us

$$
P(|Z_n - (a + b)| \geq \epsilon) \leq P(|X_n - a| \geq \epsilon/2) + P(|Y_n - b| \geq \epsilon/2) \to 0.
$$