33A/1 Linear Algebra: Practice Midterm Exam 2

Version A

Name:

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Question 1

(a) [3 points] Explain what it means for two $n \times n$ matrices $A$ and $B$ to be similar.

(b) [3 points] Show that if $A$ is similar to $B$, then $A^k$ is similar to $B^k$, for all integers $k \geq 1$.

(c) [4 points] Show that if $\vec{v}$ is an eigenvector of a matrix $A$ with $A\vec{v} = \lambda_A \vec{v}$, and $B$ with $B\vec{v} = \lambda_B \vec{v}$, then $\vec{v}$ is also an eigenvector of $A^2$, $A + B$ and $B^{-1}$ (assuming $B$ is invertible). Give the respective eigenvalues.

Answer

(a) [2 points] $A$ and $B$ are similar if there exists an invertible $n \times n$ matrix $S$ such that $B = S^{-1}AS$.

(b) [2 points] If $A$ and $B$ are similar, then $B = S^{-1}AS$ for some matrix $S$. So, $B^k = (S^{-1}AS)^k = (S^{-1}AS)(S^{-1}AS)...(S^{-1}AS) = S^{-1}AS...S^{-1}AS = S^{-1}A^nS$. Therefore $B^k$ is similar to $A^k$ under the same $S$.

(c) [3 points] $A^2\vec{v} = A\lambda_A \vec{v} = \lambda_A^2 \vec{v}$, so $\vec{v}$ is an eigenvector of $A^2$ with eigenvalue $\lambda_A^2$.

$(A + B)\vec{v} = A\vec{v} + B\vec{v} = \lambda_A \vec{v} + \lambda_B \vec{v} = (\lambda_A + \lambda_B)\vec{v}$, so $\vec{v}$ is an eigenvector of $A + B$ with eigenvalue $\lambda_A + \lambda_B$.

$\vec{v} = I_n \vec{v} = B^{-1}B\vec{v} = B^{-1}\lambda_B \vec{v}$, so $B^{-1}\vec{v} = \frac{1}{\lambda_B} \vec{v}$, so $\vec{v}$ is an eigenvector of $B^{-1}$ with eigenvalue $\lambda_B^{-1}$. 
**Question 2**

Let \( A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \).

(a) [2 points] List the eigenvalues of \( A \) and their algebraic multiplicities.

(b) [3 points] Find the eigenspaces and geometric multiplicities of the eigenvalues of \( A \).

(c) [2 points] Use your answers from (a) and (b) to decide whether \( A \) is diagonalizable.

(d) [3 points] Find a diagonalizable matrix \( B \) that differs from \( A \) in one entry only. *Hint: change an entry on the diagonal.*

**Answer**

(a) [2 points] \( A \) is a triangular matrix, so we can read the eigenvalues and their algebraic multiplicities off of the diagonal. Eigenvalues are 1 and 2, and \( \text{almu}(1) = 1 \) and \( \text{almu}(2) = 2 \).

(b) [3 points]

\[
E_1 = \ker(A - I_n) = \ker \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]

\[
E_2 = \ker(A - 2I_n) = \ker \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.
\]

So, \( \text{gemu}(1) = 1 \) and \( \text{gemu}(2) = 1 \).

(c) [2 points] The geometric multiplicities are not equal to the algebraic multiplicities for all eigenvalues, so \( A \) is not diagonalizable.

(d) [3 points]

\[
B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.
\]

\[
C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

\( B \) has all distinct eigenvalues, so the \( \text{almu} \)'s and \( \text{gemu} \)'s are 1 for all of them.
Question 3

Ben cycles to university on some days and walks on other days. If he cycles on one day, then the probability that he cycles again on the next day is 90%. Otherwise he walks. If he walks on one day, then the probability of cycling or walking the next day is 50%-50%. A transition diagram is drawn below.

(a) [3 points] Give the transition matrix $A$ that corresponds to this process.

(b) [3 points] If $\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ means that Ben cycles to the university on Monday, then $\vec{x}_1 = A\vec{x}_0 = \begin{pmatrix} .9 \\ .1 \end{pmatrix}$ is the probability distribution for Ben’s mode of transport for Tuesday. This means that he cycles with probability .9 and walks with probability .1. Find $\vec{x}_2$, the probability distribution for walking/cycling on Wednesday.

(c) [4 points] Show that, on average, Ben cycles 5 days out of 6. *Hint: this is the steady state of the distribution, or $\vec{x}_n$ as $n \to \infty$.*

Answer

(a) [3 points] $A = \begin{pmatrix} .9 & .5 \\ .1 & .5 \end{pmatrix}$

(b) [3 points] $\vec{x}_2 = A\vec{x}_1 = \begin{pmatrix} .86 \\ .14 \end{pmatrix}$.

(c) [4 points] We must show that the vector $\vec{x} = \begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix}$ is an eigenvector of $A$ with eigenvalue 1 (that is what the process $\vec{x}_n$ of the transition matrix converges to). Check:

$$A\vec{x} = \begin{pmatrix} .9 & .5 \\ .1 & .5 \end{pmatrix} \begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix} = \vec{x}.$$