Topic 4: Transforms

(a) [5 points] Let \( X \) be continuous random variable with PDF

\[
f_X(x) = \begin{cases} 
\frac{1}{3}, & 0 \leq x < 1, \\
\frac{2}{3}, & 1 \leq x < 2, \\
0, & \text{otherwise}.
\end{cases}
\]

Find the MGF of \( X \).

(b) [5 points] Let \( Y \) be a nonnegative random variable with MGF

\[
M_Y(s) = \frac{1}{2} + \frac{e^s}{2 - e^s}.
\]

Find \( P(Y = 0) \) and \( \text{var}(Y) \).

Topic 5: Sum of a Random Number of Independent RVs

(a) [5 points] Show that, if \( Y = X_1 + X_2 + \ldots + X_N \), with \( X_i \) s i.i.d., and \( N, X_1, X_2, \ldots \) independent, then, \( \text{var}(Y) = \mathbb{E}(X_i)^2 \text{var}(N) + \mathbb{E}(N) \text{var}(X_i) \).

(b) [5 points] Find the moment generating function of a geometric (parameter \( q \)) number of independent Bernoulli (parameter \( p \)) random variables.

Topic 6: Bounds

(a) [5 points] Show how Chebyshev’s Inequality follows from Markov’s Inequality.

(b) [5 points] A computer emits a binary sequence of the form \( X_1X_2X_3\ldots X_{50} \). The variables \( X_i \) are iid Bernoulli(\( \frac{1}{10} \)) random variables. Use the Union Bound to bound the probability that the sequence contains two 1s in a row.
Topic 7: Weak Law of Large Numbers

We would like to estimate the average weight \( w \) of Braeburn apples. Let \( X \) be the weight of a randomly chosen apple. We know that no apple ever weighs more than .3 kg. This gives us a bound on the variance of \( X \), given by \( \text{var}(X) = \sigma^2 \leq 22.5 \). We sample \( n \) apples, and obtain a sample average weight \( W_n \).

(a) [5 points] Give the mean and a bound on the variance of \( W_n \) in terms of \( w \) and the bound we have on the variance of \( X \).

(b) [5 points] Find the smallest \( n \) needed to obtain a value of \( W_n \) which is within .01 kg of \( w \) with .95 certainty.

Topic 8: Convergence in probability

(a) [5 points] Suppose \( X_n \overset{p}{\to} a \) and \( Y_n \overset{p}{\to} b \). Does the random variable \( Z_n = X_n - Y_n \) converge in probability to a constant?

(b) [5 points] Suppose \( X_n \overset{p}{\to} a \) and \( Y_n \overset{p}{\to} b \). Does the random variable \( V_n = \min(X_n, Y_n) \) converge in probability to a constant? (Assume that \( a < b \).)

Topic 9: Central Limit Theorem

On an exam, David scores 0, 1, 2, \ldots, 10 points, with equal probability, independently, for each of 10 questions. (So, his total score is between 0 and 100.)

(a) [6 points] David passes the exam if he scores at least 60 points. Use the central limit theorem to approximate the probability that David passes. You may write this in terms of \( \Phi(.) \) (the CDF of the standard normal) without further evaluation.

(b) [5 points] We can also say that David passes each question if he scores at least a 6 on the question. Let \( X_i \) be the indicator random variable for the event that David passes question \( i \). Now, use a De Moivre - Laplace approximation to approximate the probability that David passes exactly half of the questions.