The Kesten-Stigum Reconstruction Bound is Tight for Roughly Symmetric Channels

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Broadcasting on Trees

- **Other Interpretations:**
  - Spin system, Evolutionary tree
- **Markov Model on a Tree:**
  - b-ary Tree: T = (V,E)
  - Node states:

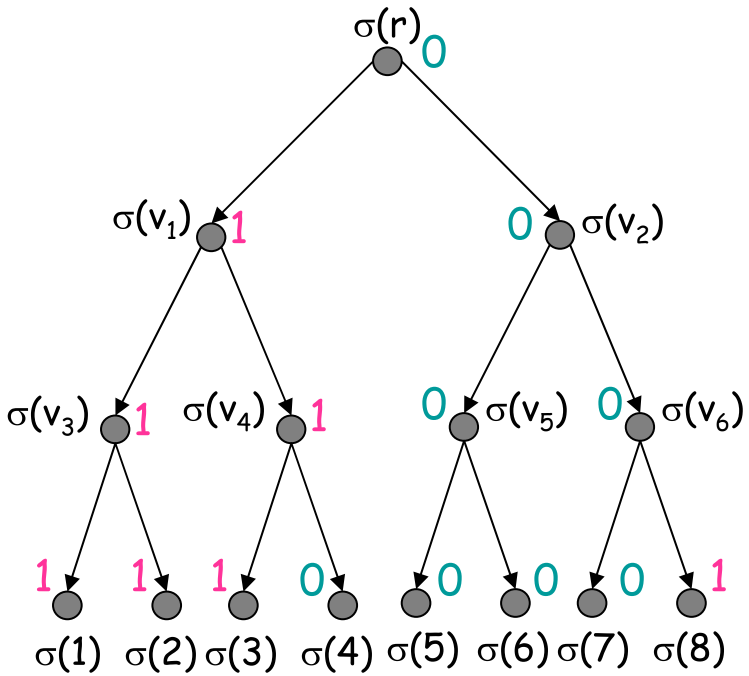
$$\{\sigma(v) \in \{0,1\} : v \in V\}$$

  - Transition Matrix:

$$M = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1+\theta & 1-\theta \\ 1-\theta & 1+\theta \end{pmatrix} + \delta \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right]$$

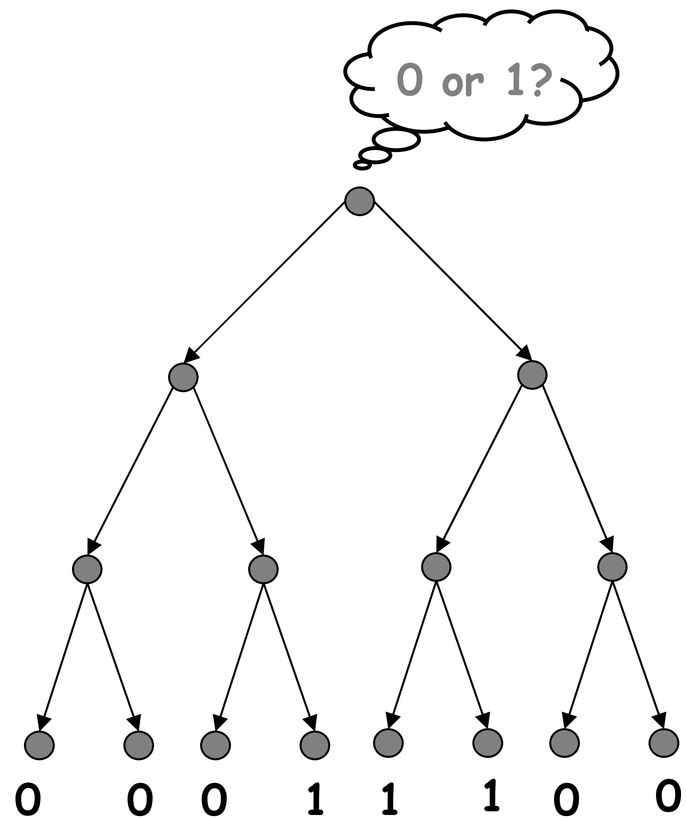
$$\theta = \lambda_2(M)$$



Reconstruction Problem

- **Root Reconstruction:**
  - *Given:* state at leaves
  - *Task:* infer state at root
- **Phase Transition:**
  - Trade-off between noise and duplication?
  - *Our main result*: for ``small asymmetry'' the tradeoff is

$$b\theta^2 = 1$$



Outline

- **Part I: Definitions and Main Result**
  - **Broadcasting Model & Reconstruction Problem**
  - **Applications in Phylogenetics & Mixing**
  - **Previous Results**
  - **Statement of New Result**

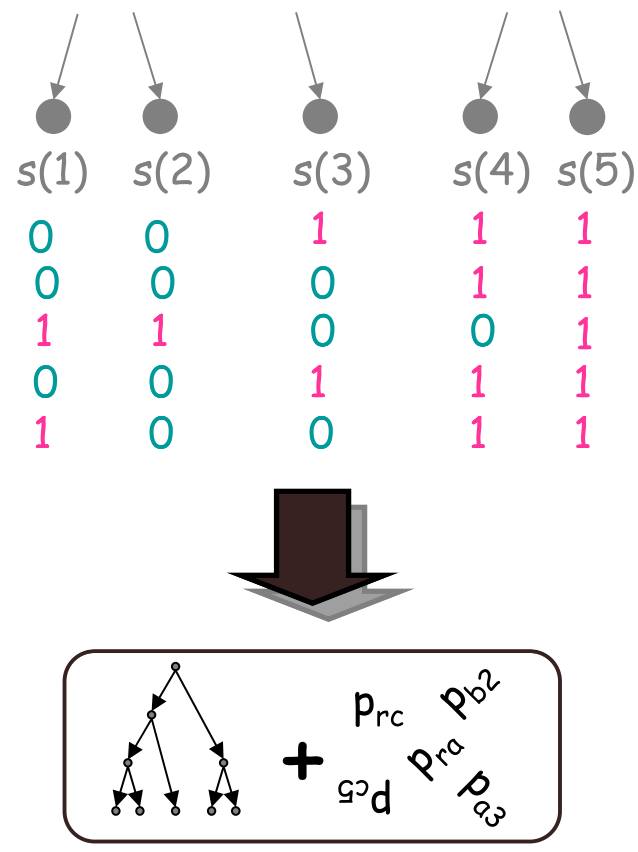
- **Part II: Proof Sketch**
  - **Magnetization**
  - **Basic Operations**
  - **Recursions**

- **Part III: Open Problems**

Phylogenetic Reconstruction Problem

- **Reconstruction:**
  - *Given:* i.i.d. samples at the leaves
  - *Task:* fully reconstruct the model, i.e. find **tree** *and* **mutation probabilities** (and, *if possible*, do so **efficiently**)

- **Previous Work:**
  - *Biology:* [Felsenstein'04]
  - *TCS (Learning):* [Ambainis-Desper-Farach-Kannan'97], [Farach-Kannan'96], [Cryan-Goldberg-Goldberg'02], [Mossel-R'05]
  - *Combinatorial Phylogeny:* [Erdos-Steel-Szekely-Warnow'97, '98], [Mossel'04a]



Phase Transition in Phylogeny: Symmetric Case ($\delta=0$)

b=2

Reconstruction **Phylogeny**

Reconstruction ⟺ $k = O(\log n)$

[Daskalakis-Mossel-Roch'06]

$2\theta^2 = 1$

No Reconstruction ⟺ $k = poly(n)$

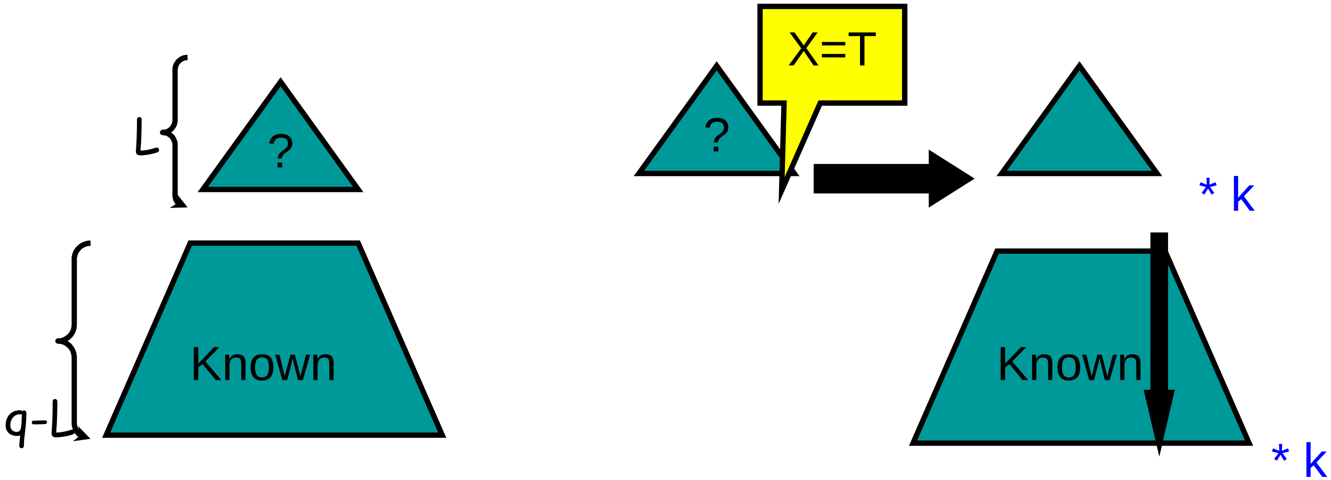
[Mossel'04]

We extend this result to roughly symmetric case

k = #samples
n = #species

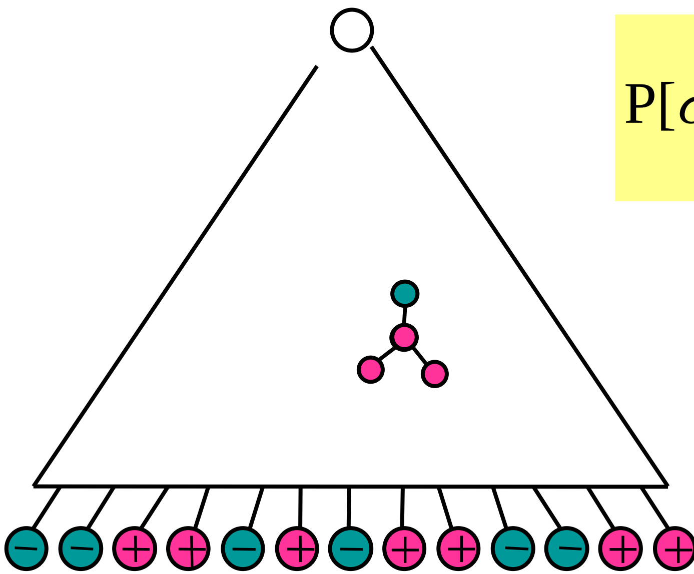
Polynomial Lower Bound at High Mutation Rates

- **Proof:**



- **Mutual Information:** I(X,Y) = H(X) - H(X | Y)
- **Data Processing Lemma:** If X and Z are cond. indep. given Y then I(X,Y) ≥ I(X,Z)

Markov Chain Monte Carlo

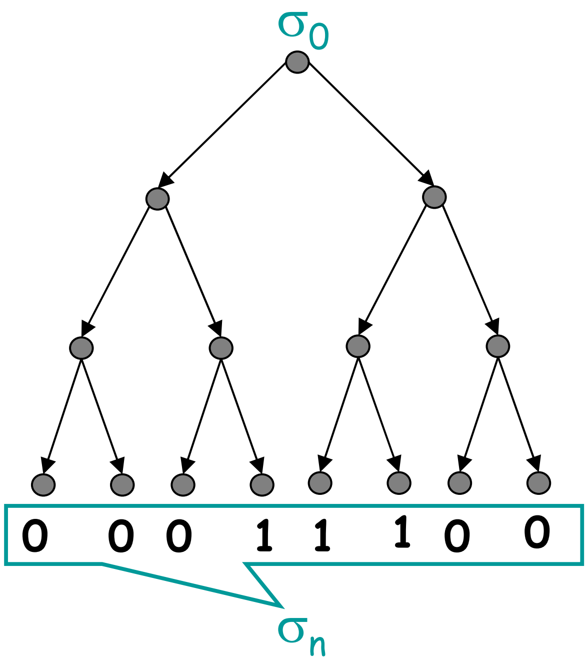


$$P[\sigma] = Z^{-1}\exp\left(\beta \sum_{(u,v)\in E}\sigma_u\sigma_v + h\sum_{v\in V}\sigma_v\right)$$

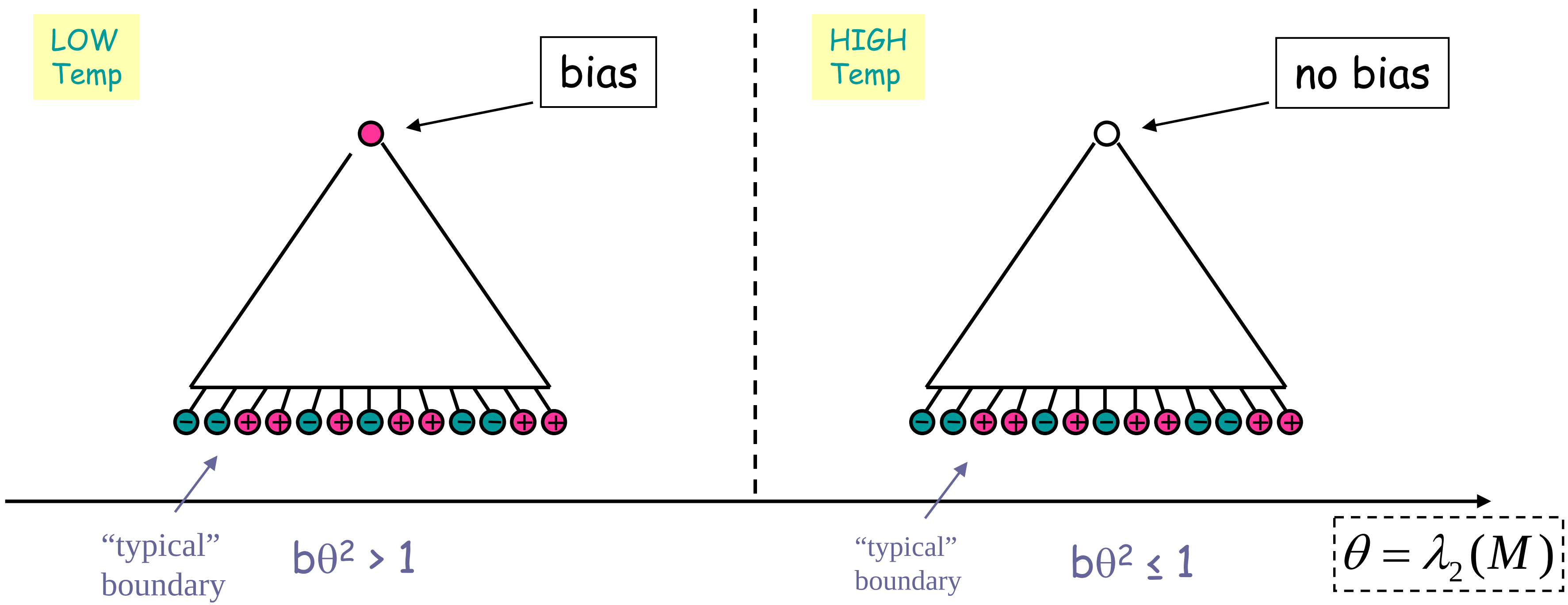
- [Martinelli-Sinclair-Weitz’04] give condition for relaxation time $O(n)$ v. $O(n^{1+\varepsilon})$
- Our results imply $b\theta^2 = 1$ is threshold for small external field (free boundary conditions)

Reconstruction Solvability

- Let T be an infinite rooted tree and T_n denote the first n levels of T.
- We say that the **reconstruction problem is solvable** if *one* of the following equivalent conditions hold:
  - $\lim_n d_{TV}(P_n^0, P_n^1) > 0$, where P_n^j denotes the distribution of σ_n conditional on $\sigma_0 = j$.

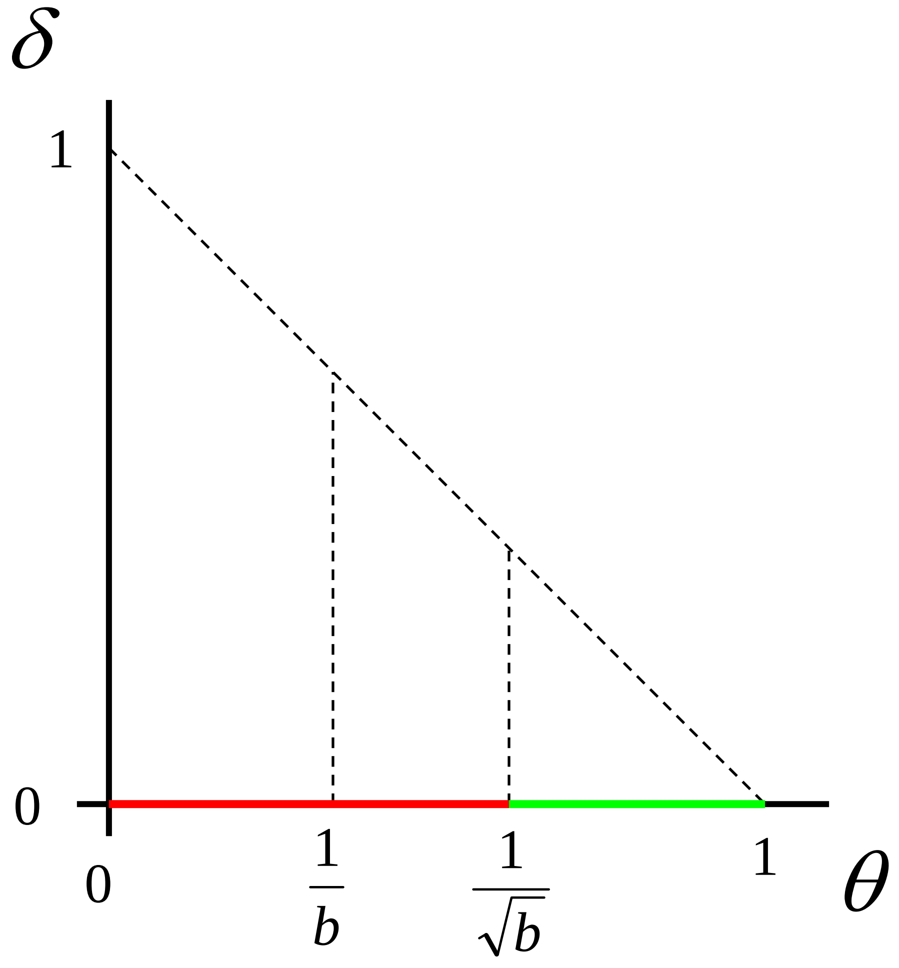


Previous Results I: Binary Symmetric Case ($\delta=0$)



The transition at $b\theta^2 = 1$ was proved by:
[Bleher-Ruiz-Zagrebnov’95], [Ioffe’96],[Evans-Kenyon-Peres-Schulman’00], [Kenyon-Mossel-Peres’01],[Martinelli-Sinclair-Weitz’04], [Borgs-Chayes-Mossel-R’06]. Solvability for $b\theta^2 > 1$ was first proved by [Higuchi’77] (and [Kesten-Stigum’66]). (Also, “spin-glass” case studied by [Chayes-Chayes-Sethna-Thouless’86].)

Phase Diagram:
Binary Channel on Binary Tree



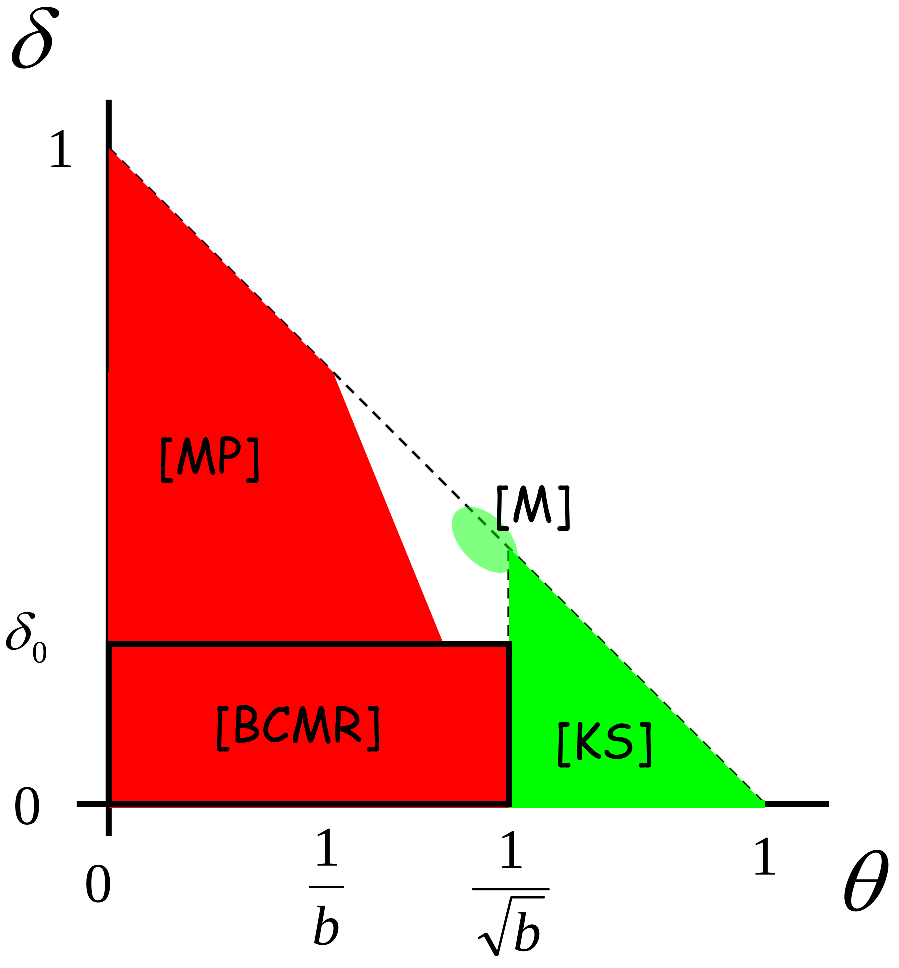
Previous Results II: Asymmetric Channels

- **Asymmetric Channels – Census Reconstruction:**
  - M, binary asymmetric channel ; θ, 2nd eigenvalue of M
  - [Kesten-Stigum'66] $b\theta^2 > 1$ implies (census-) reconstruction.

- **Counterexamples – ``$b\theta^2 > 1$ is not tight in general'':**
  - [Mossel'01] Exists asymmetric channel with $b\theta^2 < 1$ s.t. reconstruction problem is solvable
  - [Mossel, Peres'03] But reconstruction is unsolvable if:

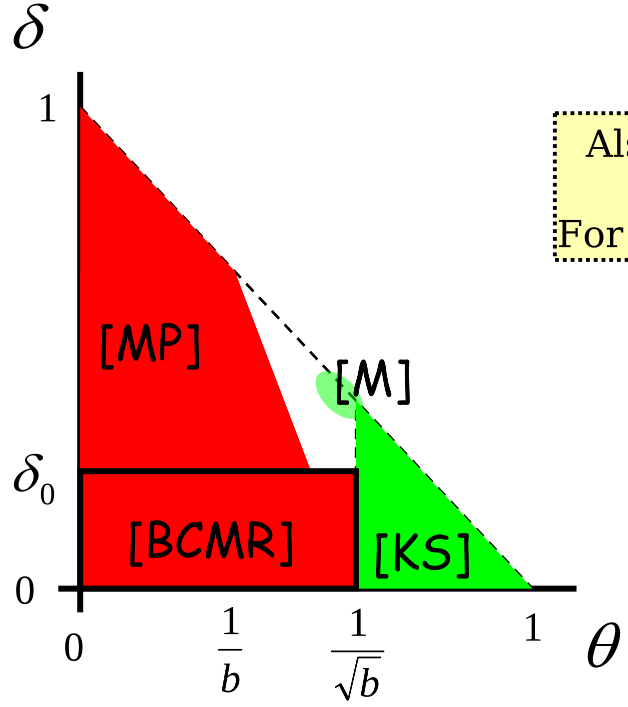
$$\frac{b\theta^2}{1-|\delta|} \leq 1$$

Phase Diagram: Binary Channel on Binary Tree



Our Result

- **Theorem** [BCMR'06]: Exists $\delta_0 > 0$ s.t. if $b\theta^2 \leq 1$ and $|\delta| < \delta_0$ then the reconstruction problem is *not solvable*.



Also, short proof

For symmetric case

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Magnetization of the Root

- We use {+1, -1}.
- Stationary distribution

$$\pi = (\pi_+, \pi_-)$$

- Basic quantity, **magnetization of the root**

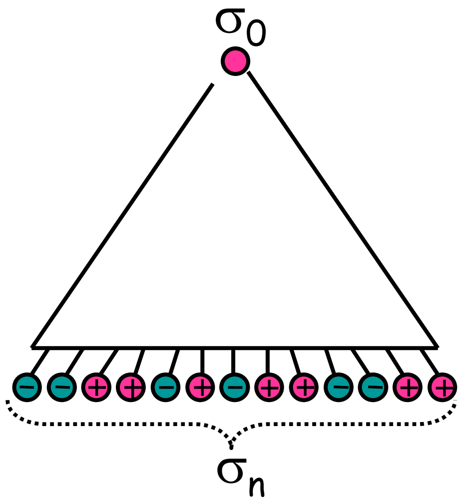
$$X_n(\sigma_n) = \pi_-^{-1}\{\pi_- \, \mathrm{P}[\sigma_0 = +1 \mid \sigma_n] - \pi_+ \, \mathrm{P}[\sigma_0 = -1 \mid \sigma_n]\}$$

- It suffices to show:

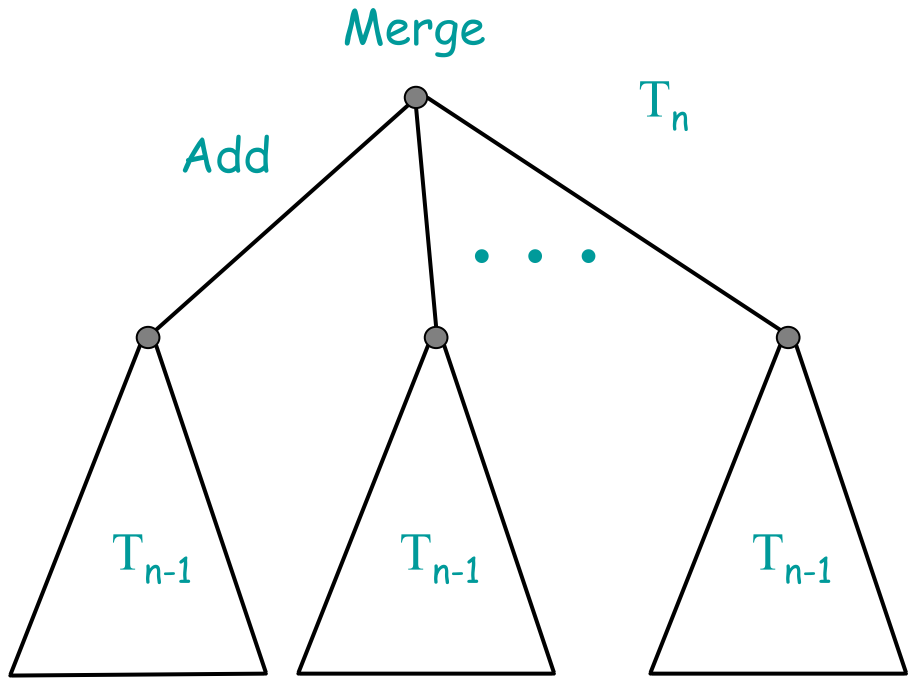
$$\bar{x}_n \equiv \mathrm{E}[X_n^2] \to 0$$

- Basic Idea: Moment Recursion

$$\bar{x}_n \le b\theta^2 \bar{x}_{n-1}$$



Recursions I: from T_{n-1} to T_n



Recursions II: Adding an Edge

$$\hat{Y} = \pi_+ \sum_{\gamma=+,-} \gamma \frac{P_{\hat{T}}[\gamma \,|\, \sigma'']}{\pi_\gamma}$$

DEF

$$= \pi_+ \sum_{\gamma=+,-} \gamma \frac{P_{\hat{T}}[\sigma'' \,|\, \gamma]}{P_{\hat{T}}[\sigma'']}$$

BAYES

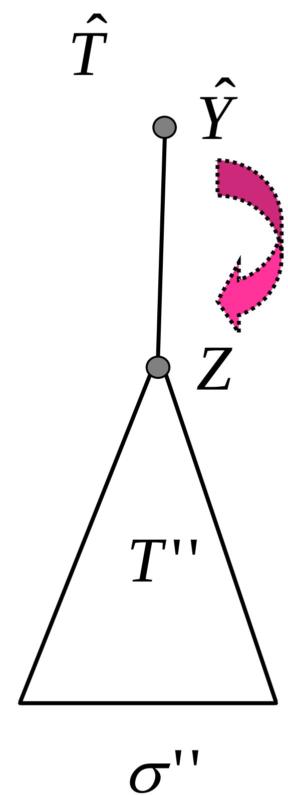
$$= \pi_+ \sum_{\gamma=+,-} \gamma \sum_{\gamma'=+,-} p_{\gamma,\gamma'} \frac{P_{T''}[\sigma'' \,|\, \gamma']}{P_{T''}[\sigma'']}$$

MARKOV

$$= \ldots$$

$$= \theta Z$$

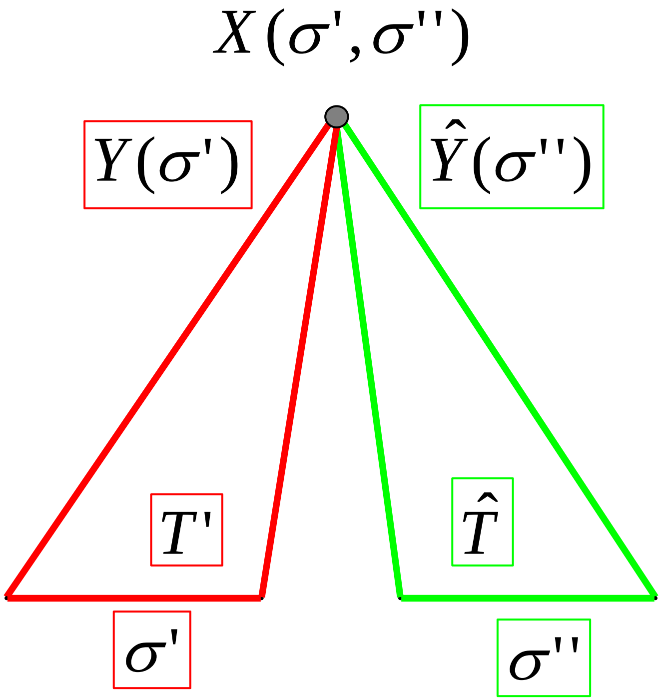
$$\hat{Y} = \theta Z$$



Recursions III: Merging Trees

Again, from BAYES and MARKOV:

$$X = \frac{Y + \hat{Y} + (\pi_{-}\pi_{+}^{-1} - 1)Y\hat{Y}}{1 + \pi_{-}\pi_{+}^{-1}Y\hat{Y}}$$



Add-Merge: Expansion

- Using

$$\frac{1}{1+r} = 1 - r + \frac{r^2}{1+r}$$

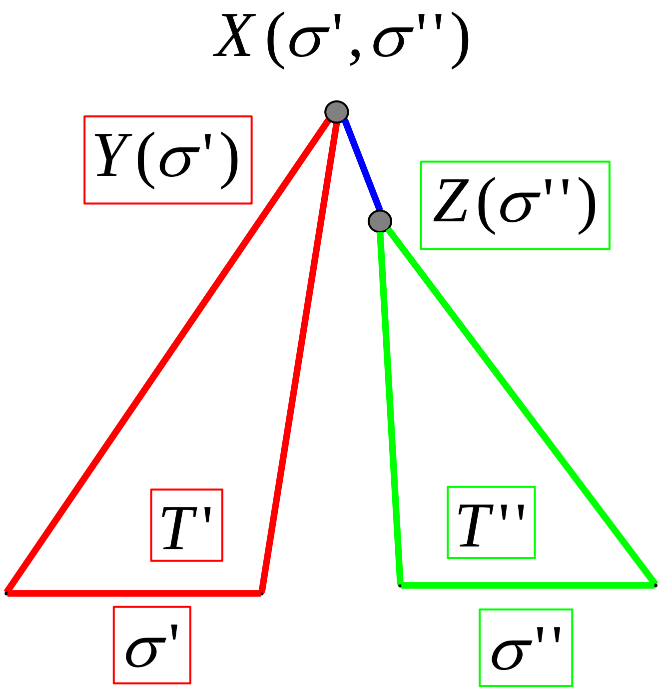
- We get

$$X = \frac{Y + \theta Z + (\pi_- \pi_+^{-1} - 1)\theta YZ}{1 + \pi_- \pi_+^{-1} \theta YZ}$$

$$\leq Y + \theta Z + (\pi_- \pi_+^{-1} - 1)\theta YZ$$

$$- \pi_- \pi_+^{-1} \theta YZ [Y + \theta Z + (\pi_- \pi_+^{-1} - 1)\theta YZ]$$

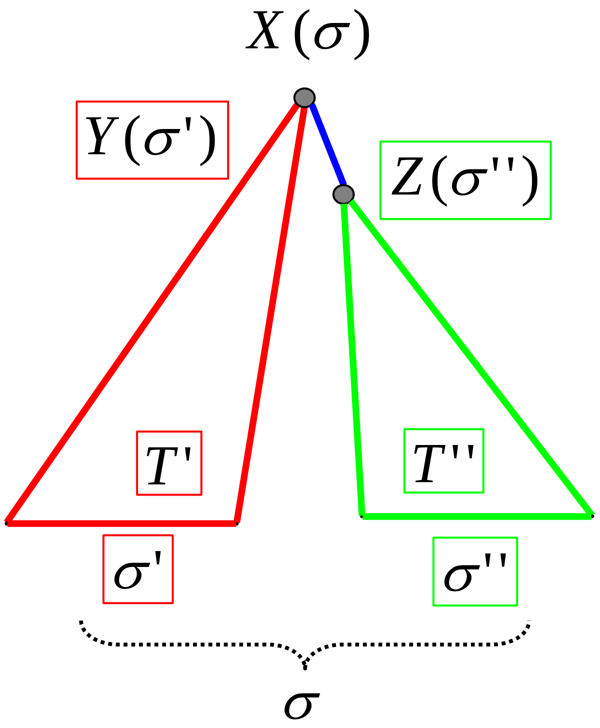
$$+ (\pi_- \pi_+^{-1} \theta YZ)^2$$



Change of Measure

$$X = \pi_-^{-1}\{\pi_- \mathrm{P}_T[+1|\sigma] - \pi_+ \mathrm{P}_T[-1|\sigma]\}$$
$$= \pi_-^{-1}\pi_+\left[\frac{\mathrm{P}_T[+1|\sigma]}{\pi_+} - 1\right]$$
$$= \pi_-^{-1}\pi_+\left[\frac{\mathrm{P}_T^+[\sigma]}{\mathrm{P}_T[\sigma]} - 1\right]$$

$$\frac{d\mathrm{P}_T^+}{d\mathrm{P}_T} = 1 + \pi_-\pi_+^{-1}X$$



$$\mathrm{E}_T^+[X] = \mathrm{E}_T[X(1+\pi_-\pi_+^{-1}X)] = \pi_-\pi_+^{-1}\,\mathrm{E}_T[X^2]$$

Add-Merge: Basic Inequality

- Recall

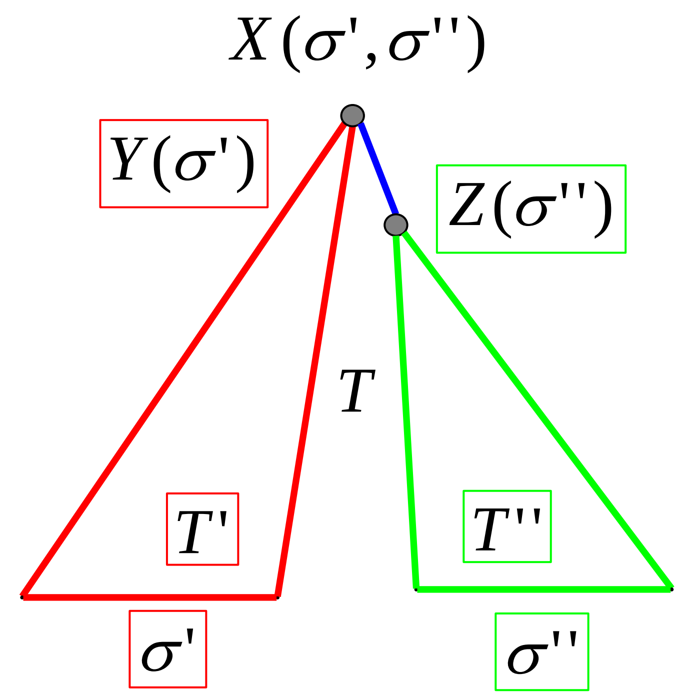
$$X \le Y + \theta Z + (\pi_- \pi_+^{-1} - 1)\theta YZ + (\pi_- \pi_+^{-1} \theta YZ)^2$$
$$- \pi_- \pi_+^{-1} \theta YZ[Y + \theta Z + (\pi_- \pi_+^{-1} - 1)\theta YZ]$$

- Taking expectation, after some work we get (as long as |δ| is small enough)

$$\bar{x} \le \bar{y} + \theta^2 \bar{z}$$

- Repeating b times

$$\bar{x}_n \le b\theta^2 \bar{x}_{n-1}$$



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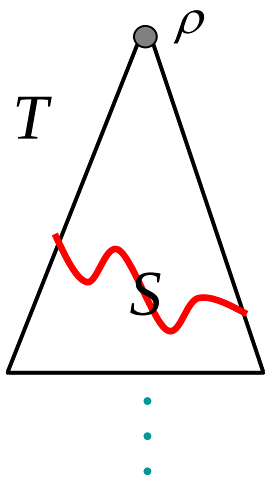
Branching Number

- **General Trees:**
  - Let T be an infinite tree and θ a function on the edges
  - **Branching number**:

$$\mathrm{br}(T,\theta) = \inf\left\{\lambda > 0 : \inf_{\text{cutsets } S} \sum_{x\in S}\left(\lambda^{-|x|}\prod_{e\in \mathrm{path}(\rho,x)}\theta^2(e)\right) = 0\right\}$$

  - [Lyons'90] $\mathrm{br}(T,1)^{-1}$ is the critical threshold for percolation on trees
  - Taking $\theta^2(e) = p(e)$, the threshold for percolation is $\mathrm{br}(T,\theta) = 1$



More General Result

- **General Trees – Previous Results:**
  - Branching number:

$$\mathrm{br}(T,\theta) = \inf\left\{\lambda > 0 : \inf_{\text{cutsets } S} \sum_{x\in S}\left(\lambda^{-|x|}\prod_{e\in \mathrm{path}(\rho,x)}\theta^2(e)\right) = 0\right\}$$

  - [Evans-Kenyon-Peres-Schulman'00] Binary symmetric case on general tree, solvable iff br(T, θ) > 1.

- **Theorem** [BCMR'06]: Let $0 \le \theta_0 < 1$. Exists $\delta_0 > 0$ s.t.
  - For all stationary distributions $\pi=(\pi_+, \pi_-)$ with $\max\{|\delta(\pi,\theta)|, |\delta(\pi,-\theta)|\} < \delta_0$
  - For all trees with $\sup_e|\theta(e)| \le \theta_0$ and $\mathrm{br}(T,\theta) \le 1$

The reconstruction problem is *not solvable*

More General Result: Proof Sketch

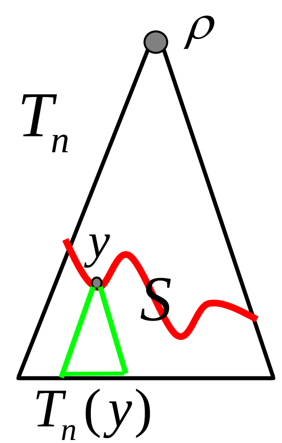
- Fix ε > 0
- From

$$\mathrm{br}(T,\theta) = \inf\left\{\lambda > 0 : \inf_{\text{cutsets } S} \sum_{x\in S}\left(\lambda^{-|x|}\prod_{e\in \mathrm{path}(\rho,x)}\theta^2(e)\right) = 0\right\} \le 1$$

we can find (minimal) cutset S s.t.

$$\sum_{x\in S}\prod_{e\in \mathrm{path}(\rho,x)}\theta^2(e) \le \varepsilon$$

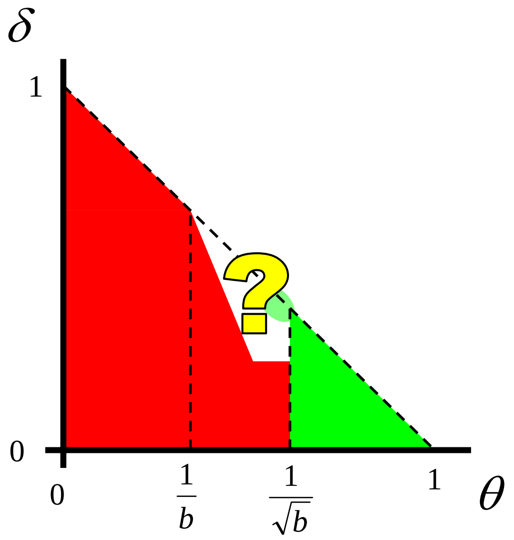
- Take n large enough
- Apply basic inequality repeatedly from the root down to S to get



$$\bar{x}_n \le \sum_{y\in S} \mathrm{E}_{T_n(y)}[Y^2] \prod_{e\in \mathrm{path}(\rho,y)}\theta^2(e) \le \sum_{y\in S}\prod_{e\in \mathrm{path}(\rho,y)}\theta^2(e) \le \varepsilon$$

Open Problems

- Complete phase diagram for binary asymmetric channels?



- Beyond two-state case?
- Connection with spinglasses?