HW4 Solutions

Sec 2.2 ② (b) $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$

⑦ Suppose $\beta = (v_1, \dots, v_n)$.
If $x = \sum_{i=1}^n a_i v_i$ and $y = \sum_{i=1}^n b_i v_i$
then $cx+y = c\sum_{i=1}^n a_i v_i + \sum_{i=1}^n b_i v_i = \sum_{i=1}^n (ca_i+b_i) v_i$,
and hence $T(cx+y) = (ca_1+b_1, \dots, ca_n+b_n)^t$
$= c(a_1, \dots, a_n)^t + (b_1, \dots, b_n)^t$
$= cT(x) + T(y)$.
$\therefore$ T is linear

⑩ Let α be a basis for W $(\#\alpha = k)$
and extend α to a basis β for V $(\#\beta = n, \beta \supset \alpha)$.
write $\alpha = (v_1, \dots, v_k)$, $\beta = (v_1, \dots, v_n)$.
Then since $T(W) \subset W$,
$T(v_j) \in \operatorname{span} \alpha$ for $1 \le j \le k$
ie $T(v_j) = \sum_{i=1}^k A_{ij} v_i$ for $1 \le j \le k$.
Thus $[T]_\beta$ is of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$

Sec 2.3 ⑦ We will prove the following generalization of Thm 2.10:
Suppose V_i are v-spaces and $T_i, T_i' : V_i \to V_{i+1}$ linear
Then ① $T_2(T_1 + T_1') = T_2T_1 + T_2T_1'$ and $(T_2+T_2')T_1 = T_2T_1 + T_2'T_1$
② $T_3(T_2T_1) = (T_3T_2)T_1$
③ $T_1 I_{V_1} = T_1 = I_{V_2} T_1$
④ $a(T_2T_1) = (aT_2)T_1 = T_2(aT_1)$ for $a \in F$

Pf: ① $(T_2(T_1+T_1'))(x) = T_2((T_1+T_1')(x))$

$= T_2(T_1(x)+T_1'(x))$

$= T_2(T_1(x)) + T_2(T_1'(x))$ by linearity

$= (T_2T_1)(x) + (T_2T_1')(x)$

$= (T_2T_1 + T_2T_1')(x)$

similarly, $((T_2+T_2')T_1)(x) = (T_2T_1 + T_2'T_1)(x)$

② $(T_3(T_2T_1))(x) = T_3((T_2T_1)(x))$

$= T_3(T_2(T_1(x)))$

and $((T_3T_2)T_1)(x) = (T_3T_2)(T_1(x))$

$= T_3(T_2(T_1(x)))$

so they are equal

③ $(T_1 I_{V_1})(x) = T_1(I_{V_1}(x)) = T_1(x)$

$(I_{V_2}T_1)(x) = I_{V_2}(T_1(x)) = T_1(x)$

④ $(a(T_2T_1))(x) = a((T_2T_1)(x))$

$= a(T_2(T_1(x)))$

$= (aT_2)(T_1(x))$

$= ((aT_2)T_1)(x)$

similarly, $(a(T_2T_1))(x) = a(T_2(T_1(x)))$

$= T_2(a(T_1(x)))$ by linearity

$= T_2((aT_1)(x))$

$= (T_2(aT_1))(x)$

13 (a) Let $z = (a_1, \dots, a_p)^t \in F^p$

then $z = \sum_{j=1}^{p} a_j e_j$.

Let $B = (v_1, \dots, v_p) \in M_{n \times p}(F)$

then by Thm 2.13b, $v_j = Be_j$.

Thus by linearity, $Bz = \sum_{j=1}^{p} a_j Be_j = \sum_{j=1}^{p} a_j v_j$

(b) Let $A = (w_1, \dots, w_n) \in M_{m \times n}(F)$

and write $AB = (u_1, \dots, u_p) \in M_{m \times p}(F)$.

Then by Thm 2.13a, $u_j = Av_j$,

and by (a) above, $Av_j = \sum_{k=1}^{n} (v_j)_k w_k$,

so $u_j = \sum_{k=1}^{n} (v_j)_k w_k$.

(c) Let $w = (a_1, \dots, a_m)$ be a row vector

and $A = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in M_{m \times n}(F)$.

Then $(wA)^t = A^t w^t = (y_1^t, \dots, y_m^t) \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$

$= \sum_{j=1}^{m} a_j y_j^t$ by (a) above,

and hence $wA = \sum_{j=1}^{m} a_j y_j$

(d) Let $B = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in M_{n \times p}(F)$

and write $AB = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in M_{m \times n}(F)$.

Then $(x_1^t, \dots, x_m^t) = (AB)^t = B^t A^t = (z_1^t, \dots, z_n^t)(y_1^t, \dots, y_m^t)$

and hence $x_i^t = \sum_{k=1}^{n} (y_i^t)_k z_k^t$ by (b) above.

Therefore $x_i = \sum_{k=1}^{n} (y_i^t)_k z_k = \sum_{k=1}^{n} (y_i)_k z_k$.