THEOREMS AND CONJECTURES ON THE DROPLET-DRIVEN RELAXATION OF STOCHASTIC ISING MODELS

ROBERTO H. SCHONMANN*
Department of Mathematics
University of California
Los Angeles, CA 90024
U.S.A.

Abstract. Recent rigorous results on droplet-driven relaxation of stochastic Ising models in the vicinity of the phase transition region are reviewed. Further conjectures are raised based on the same sort of heuristic picture which suggested the proven results, and some new results on these lines are announced and some of their proofs sketched.

Key words: Stochastic Ising model, Glauber dynamics, relaxation, droplets, metastability, spectral gap.

1. Introduction

In this note I will summarize results proven recently in [Sch3], on the relaxation mechanism of stochastic Ising models (also known as Glauber dynamics or kinetic Ising models) in the proximity of the phase transition region. I will also explain how these results were predicted by a heuristic type of reasoning and, more importantly, how the same sort of heuristics can be used to raise further conjectures on the behavior of the model. Surprisingly, it seems to me that a proper understanding of the picture which emerges from this heuristic view-point, including the rich diversity of behaviors (depending on the parameters volume, temperature and external magnetic field) is a novel contribution to the theory, even at the non-rigorous level.

The paper is being written having in mind that it may be read by people with different backgrounds in the areas of statistical mechanics and interacting particle systems. For those who are less familiar with these subjects, I recommend reading at this moment Section 2, and then returning to the next paragraph. For the experts, Section 2 will play mostly the role of a reference for notation, and can be consulted later.

We will consider the basic Ising model on $\mathbb{Z}^d$, with formal Hamiltonian

$$H_h(\sigma) = -\frac{1}{2} \sum_{x,y \text{ n.n.}} \sigma(x)\sigma(y) - \frac{h}{2} \sum_{x} \sigma(x), \tag{1}$$

where $\sigma(x) = \pm 1$ is the spin at the site $x \in \mathbb{Z}^d$, and the first sum runs over pairs of sites which are nearest neighbors in $\mathbb{Z}^d$, each pair counted only once. The time evolution is introduced as a spin flip Markov process which is reversible with respect

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to the corresponding Gibbs measures at temperature T. The flip rates will be supposed to satisfy certain regularity conditions, but these will be very mild, so that essentially all common choices for these rates will be considered.

We are concerned with the relaxation mechanism of these systems in the vicinity of the phase transition region, i.e., at low temperature and under small but non-null external field h.

These systems that we are considering are probably the favorite model systems for investigators addressing the issue of relaxation to equilibrium of systems close to a first order phase transition. The literature on the subject is vast, because the problem is of interest to researchers in such diverse areas as metallurgy, chemistry, physics and probability. One of the conspicuous features of relaxation phenomena close to discontinuous transitions is the presence of metastable behavior, in which the system seems, for a long time, to have reached equilibrium, but in a state which is actually far from the true equilibrium state, and is close to what the equilibrium would be for values of the parameters at the other side of the transition region. A considerable number of review papers and monographs has been written on the subjects of metastability and relaxation close to transition regions. The reader may consult for instance [GD], [GSS] and [Koc] for accounts which emphasize non-rigorous results. A good review of rigorous investigations on the problem of metastability is [PL].

Many papers have been written on simulations and analytical study (rigorous and non-rigorous) of the stochastic Ising models in the regime which concerns us. The reader will find a large number of references in the reviews quoted above and a constant stream of papers on the subject in more recent issues of journals in statistical mechanics, mathematical physics and other related subjects. Different proposals have appeared on what aspects of the problem one should study, what to measure in simulations, what to compute and what to prove. Even after several decades of investigations there seems to still be a fair amount of controversy on what the most relevant aspects of the problem are, and on questions of the type: what are the mathematical theorems that should be proved (or at least conjectured) and that properly capture the experimentally-manifestly-clear metastable behavior of the systems. Just to quote a few of the different issues:

1. Can one see the metastable behavior by looking at time averages or only if looking at typical individual paths of the process?
2. Is there any type of metastable behavior for these models on the infinite lattice, or only on finite lattices?
3. Is there a clear cut definition of what a 'metastable state' is?
4. Is there a sharp value of the external field (depending possibly on the temperature) which separates a region (small $|h|$) where the answer to the previous question is yes, from one where the answer is no?
5. Is the fact that the relaxation time becomes very large when h is small reflected by the gap in the spectrum of the generator of the process vanishing as the inverse of the relaxation time as $h \searrow 0$?
6. To properly characterize and study metastability, should one consider the process conditioned to not having reached the configurations which are typical in equilibrium, until a large time?

I am quoting all these questions here just to give to the reader an idea of the richness of the area. Some of these questions are, of course, essentially a matter of the use that one wants to make of the word 'metastability', and we do not want to enter in such a discussion. On the other hand, I believe that the results reviewed and announced in this paper and the picture and the further conjectures proposed are helping to clarify some of the real problems in some of these questions.

The rest of this paper is organized as follows. In Section 2 the basic notions and notation are introduced in a fashion hopefully readable by a non-expert. In Section 3, the main result in [Sch3] is reviewed and related to what is observed in simulations. In Section 4 this result is explained on heuristic grounds. In Section 5 some rigorous counterparts of the heuristics are reviewed. In Section 6 different quantities which measure the speed of convergence to equilibrium are introduced, recent results on these quantities are described and further conjectures on their behavior are raised and heuristically motivated. In Section 7 results and conjectures on the behavior of versions of the stochastic Ising models on finite lattices, with sizes which are being scaled as $h \searrow 0$ are presented. Finally, In Section 8, we recall that interesting and usually sharper results can be proven for the same systems in regimes in which the temperature is scaled to 0, rather then the external field, or together with the external field; some new results are announced, which support the conjectures raised in the previous sections of this paper.

Readers who are familiar with the content of [Sch3] can go directly to Section 6.

2. The Models

I have tried to make everything here as standard as possible, so that readers who are familiar with the models will browse quickly through this subsection, finding few things to which they are not used (like Proposition 1). The presentation is self-contained, but most statements are made without proofs, and I refer readers to the books [Lig1] and [Rue], and other references therein for complete treatments. Some of the proofs that were omitted are relatively easy, and having newcomers in mind, I indicate them in the text as exercises (sometimes with hints).

THE LATTICE: We will consider models on the lattices $\mathbb{Z}^d$, where d is the space dimensionality. Because the dimension d will in general be arbitrary but fixed, we will omit it in most of the notation. The cardinality of a set $\Gamma \subset \mathbb{Z}^d$ will be denoted by $|\Gamma|$. The family of finite subsets of $\mathbb{Z}^d$ will be denoted by $\mathcal{F}$. For each $x \in \mathbb{Z}^d$, we define the usual norms $\|x\|_p = (|x_1|^p + \ldots + |x_d|^p)^{1/p}$, $p > 0$ finite, and $\|x\|_\infty = \max\{|x_1|, \ldots, |x_d|\}$. The interior and exterior boundaries of a set $\Gamma \subset \mathbb{Z}^d$ will be denoted, respectively by

$$\partial_{\text{int}}\Gamma := \{x \in \Gamma : \|x - y\|_1 = 1 \text{ for some } y \notin \Gamma\},$$

and

$$\partial_{\text{ext}}\Gamma := \{x \notin \Gamma : \|x - y\|_1 = 1 \text{ for some } y \in \Gamma\}.$$

For integer i, we introduce the notation

$$V_i = \{x \in \mathbb{Z}^d : \|x\|_\infty \leq i\},$$

for the box centered at the origin which has side-length $2i+1$. But because usually the side-length of such a box is of particular importance for us, we will mostly be using the alternative notation

$$\Lambda(l) = \text{ largest } V_i \text{ which has side-length not larger than } l.$$

The set of bonds, i.e., (unordered) pairs of nearest neighbors is defined as

$$\mathbb{B} = \{\{x, y\} : x, y \in \mathbb{Z}^d \text{ and } \|x - y\|_1 = 1\}.$$

Given a set $\Gamma \in \mathcal{F}$ we define also

$$\mathbb{B}_\Gamma = \{\{x, y\} : x, y \in \Gamma \text{ and } \|x - y\|_1 = 1\},$$

$$\partial\mathbb{B}_\Gamma = \{\{x, y\} : x \in \Gamma, y \notin \Gamma \text{ and } \|x - y\|_1 = 1\}.$$

A chain is a sequence of distinct sites $x_1, \ldots, x_n$, with the property that for $i = 1, \ldots, n-1$, $\{x_i, x_{i+1}\} \in \mathbb{B}$. The sites x_1 and x_n are called the end-points of the chain $x_1, \ldots, x_n$. A set of sites with the property that each two of them can be connected by a chain contained in the set is said to be a connected set.

The Configurations and Observables: At each site in $\mathbb{Z}^d$ there is a spin which can take values -1 and $+1$. The configurations will therefore be elements of the set $\{-1, +1\}^{\mathbb{Z}^d} =: \Omega$. Given $\sigma \in \Omega$, we write $\sigma(x)$ for the spin at the site $x \in \mathbb{Z}^d$. Two configurations are specially relevant: $-\underline{1}$ and $+\underline{1}$, which are, respectively, the ones with all spins -1 and $+1$. When these configurations appear as a subscript or superscript, we will usually abbreviate them by, respectively, $-$ and $+$. The single spin space, $\{-1, +1\}$ is endowed with the discrete topology and Ω is endowed with the corresponding product topology. The following definition will be important when we introduce finite systems with boundary conditions later on; given $\Gamma \in \mathcal{F}$ and a configuration $\eta \in \Omega$, we introduce

$$\Omega_{\Gamma,\eta} := \{\sigma \in \Omega : \sigma(x) = \eta(x) \text{ for all } x \notin \Gamma\}.$$

Real-valued functions with domain in Ω are called observables. For each observable f, we use the notation $||f||_\infty := \sup_{\eta \in \Omega} |f(\eta)|$. Local observables are those which depend only on the values of finitely many spins, more precisely, $f : \Omega \to \mathbb{R}$ is a local observable if there exists a set $S \in \mathcal{F}$ such that $f(\sigma) = f(\eta)$ whenever $\sigma(x) = \eta(x)$ for all $x \in S$. The smallest S with this property is called the support of f. Clearly, if f is a local observable, then $||f||_\infty < \infty$. The topology introduced above on Ω, has the nice feature that it makes the set of local observables dense in the set of all continuous observables.

On Ω the following partial order is introduced:

$$\eta \leq \zeta \quad \text{if} \quad \eta(x) \leq \zeta(x) \text{ for all } x \in \mathbb{Z}^d.$$

A particularly important role will be played in this paper by the non-decreasing local observables. Clearly every local observable is of bounded variation, and, as such, can be written as the difference between two non-decreasing ones.

A $-$chain in a configuration σ, or simply a σ-chain, is a chain of sites, $x_1, \dots, x_n$, as defined above, with the property that for each $i = 1, \dots, n$, $\sigma(x_i) = -1$. The $-$clusters in a configuration σ are the connected components of the set of sites where the spin is -1 in the configuration σ. A $-$cluster is called infinite if it contains infinitely many sites.

THE PROBABILITY MEASURES: We endow Ω also with the Borel σ-algebra corresponding to the topology introduced above. In this fashion, each probability measure μ in this space can be identified by the corresponding expected values $\int f d\mu$ of all the local observables f. A sequence of probability measures, $(\mu_n)_{n=1,2,\dots}$, is said to converge weakly to the probability measure ν in case

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\nu \ \text{ for every continuous observable } f. \tag{2}$$

The family of probability measures on Ω will be partially ordered by the following relation: $\mu \leq \nu$ if

$$\int f d\mu \leq \int f d\nu \ \text{ for every continuous non-decreasing observable } f. \tag{3}$$

Because the local observables are dense in the set of continuous observables, we can restrict ourselves to the local ones in (2) and (3). Moreover, because every local observable is the difference between two non-decreasing ones, we can also restrict ourselves to those in (2).

THE GIBBS MEASURES: We will consider always the formal Hamiltonian (1). In order to give precise definitions, we define, for each set $\Gamma \in \mathcal{F}$ and each boundary condition $\eta \in \Omega$,

$$H_{\Gamma,\eta,h}(\sigma) = -\frac{1}{2} \sum_{\{x,y\}\in\mathbb{B}_\Gamma} \sigma(x)\sigma(y) - \frac{1}{2} \sum_{\substack{\{x,y\}\in\partial\mathbb{B}_\Gamma \\ y\notin\Gamma}} \sigma(x)\eta(y) - \frac{h}{2} \sum_{x\in\Gamma} \sigma(x), \tag{4}$$

where $h \in \mathbb{R}$ is the external field and $\sigma \in \Omega$ is a generic configuration. The Gibbs (probability) measure in Γ with boundary condition η under external field h and at temperature $T = 1/\beta$ is now defined on Ω as

$$\mu_{\Gamma,\eta,h}(\sigma) = \begin{cases} \dfrac{\exp(-\beta H_{\Gamma,\eta,h}(\sigma))}{\displaystyle\sum_{\zeta\in\Omega_{\Gamma,\eta}} \exp(-\beta H_{\Gamma,\eta,h}(\zeta))} & \text{if } \sigma \in \Omega_{\Gamma,\eta}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that we omit in the notation reference to the temperature T, because it will be usually fixed. The following property is a consequence of the fact that the Hamiltonian only involves interactions between nearest neighbors: given $\Gamma \in \mathcal{F}$, if $\eta(x) = \zeta(x)$ for every $x \in \partial_{\text{ext}}\Gamma$, then

$$\int f d\mu_{\Gamma,\eta,h} = \int f d\mu_{\Gamma,\zeta,h}, \tag{5}$$

for every local observable f whose support is contained in Γ. The next property is known as the DLR equation: given $\Gamma \subset \Gamma' \in \mathcal{F}$ and a pair of configurations η and η' which are identical off Γ', we have

$$\mu_{\Gamma',\eta',h}(\,\cdot\,|\,\Omega_{\Gamma,\eta}) = \mu_{\Gamma,\eta,h}(\cdot). \tag{6}$$

The Gibbs measures satisfy the following monotonicity relations to which we will refer as the Holley–FKG inequalities.

$$\text{If } \eta \le \zeta \text{ and } h_1 \le h_2, \text{ then, for each } \Gamma \in \mathcal{F}, \;\; \mu_{\Gamma,\eta,h_1} \le \mu_{\Gamma,\zeta,h_2}.$$

A Gibbs measure for the infinite system on $\mathbb{Z}^d$ is defined now as any probability measure, μ, which satisfies the DLR equations in the sense that for every $\Gamma \in \mathcal{F}$ and μ-almost all $\eta \in \Omega$

$$\mu(\,\cdot\,|\,\Omega_{\Gamma,\eta}) = \mu_{\Gamma,\eta,h}(\cdot). \tag{7}$$

Alternatively and equivalently, Gibbs measures can be defined as elements of the closed convex hull of the set of weak limit points of sequences of the form $(\mu_{\Gamma_i,\eta_i,h})_{i=1,2,\ldots}$, where each Γ_i is finite and $\Gamma_i \to \mathbb{Z}^d$, as $i \to \infty$, in the sense that $\bigcup_{i=1}^{\infty}\bigcap_{j=i}^{\infty}\Gamma_j = \mathbb{Z}^d$. Together, (5) and the DLR equations, (6) and (7), imply the Markov property for the Gibbs measures; for instance, if μ is a Gibbs measure for the infinite system under external field h, then for arbitrary $\Gamma \in \mathcal{F}$ and μ-almost all $\eta, \zeta \in \Omega$ such that $\eta(x) = \zeta(x)$ for every $x \in \partial_{\text{ext}}\Gamma$,

$$\int f d\mu(\cdot\,|\,\Omega_{\Gamma,\eta}) = \int f d\mu(\cdot\,|\,\Omega_{\Gamma,\zeta}),$$

for every local observable f whose support is contained in Γ.

The Holley–FKG inequalities can be used to prove that for each value of T and h, $\mu_{\Lambda(l),-,h}$ (resp. $\mu_{\Lambda(l),+,h}$) converges weakly, as $l \to \infty$, to a probability measure that we will denote by $\mu_{-,h}$ (resp. $\mu_{+,h}$). (Take the last statement as an exercise; hint: consider first non-decreasing local observables.) If $h \neq 0$, or $d = 1$, it is also known that

$$\mu_{-,h} = \mu_{+,h} =: \mu_h, \tag{8}$$

while if $d \ge 2$ and $h = 0$ the same is true if the temperature is larger than a critical value $T_c > 0$, which depends on the dimension, and is false for $T < T_c$. Moreover, for the values of T and h for which (8) holds, any weak limit of any sequence of the form $(\mu_{\Gamma_i,\eta_i,h})_{i=1,2,\ldots}$, where $\mathcal{F} \ni \Gamma_i \to \mathbb{Z}^d$, is identical to μ_h. Therefore we conclude that whenever (8) holds there is a unique Gibbs measure for the infinite system. When (8) fails, there is more than one Gibbs measure for the infinite system, and we say that there is phase coexistence. We use the following abbreviations and names:

$$\begin{aligned} \mu_{-,0} &:= \mu_- = \text{ the minus phase,} \\ \mu_{+,0} &:= \mu_+ = \text{ the plus phase.} \end{aligned}$$

Another known fact is that for fixed T

$$\mu_h \to \mu_+ \text{ weakly, as } h \searrow 0, \tag{9}$$

and

$$\mu_h \to \mu_- \text{ weakly, as } h \nearrow 0. \tag{10}$$

(Proving these facts is an excellent exercise. Hints: consider non-decreasing local observables, and use the Holley–FKG inequalities. Half of each statement follows then easily. For the other half you can compare the infinite system to a finite one using again Holley–FKG, then use 'continuity' of the Gibbs measure as a function of the external field in a finite box, and finally let the size of the box grow.) For the expected value corresponding to a Gibbs measure $\mu_{...}$, in finite or infinite volume, we will use the notation

$$\langle f \rangle_{...} := \int f d\mu_{...},$$

where ... stands for arbitrary subscripts. The spontaneous magnetization at temperature T is defined as

$$m^*(T) = \langle \sigma(0) \rangle_+.$$

(Here we are using a common and convenient form of abuse of notation: $\sigma(x)$ is being used to denote the observable which associates to each configuration the value of the spin at the site x in that configuration.) It is known that $m^*(T) > 0$ if and only if $\mu_- \neq \mu_+$, and also that $\lim_{T \searrow 0} m^*(T) = 1$.

THE DYNAMICS: We introduce now for the Ising model above, the time evolution known as stochastic Ising model or Glauber dynamics. First we recall that a spin flip system is defined as a Markov process on the state space Ω, whose generator, L, acts on a generic local observable f as

$$(Lf)(\sigma) = \sum_{x \in \mathbb{Z}^d} c(x,\sigma)(f(\sigma^x) - f(\sigma)), \tag{11}$$

where σ^x is the configuration obtained from σ by flipping the spin at the site x, and $c(x,\sigma)$ is called the rate of flip of the spin at the site x when the system is in the state σ. In order for this generator to be well defined and indeed generate a unique Markov process, one has to assume that the rates $c(x,\sigma)$ satisfy certain regularity conditions. For our purposes here, we will actually restrict ourselves to the following conditions, which are more than enough to assure the existence and uniqueness of the process.

(H1) (*Translation invariance*) For every $x, y \in \mathbb{Z}^d$,

$$c(x,\sigma) = c(x+y, \theta_y \sigma),$$

where $\theta_y \sigma$ is the configuration obtained by shifting σ by y, i.e., $(\theta_y \sigma)(z) = \sigma(z-y)$.

(H2) (*Finite range*) There exists R such that

$$c(0,\eta) = c(0,\zeta) \quad \text{if} \quad \eta(x) = \zeta(x) \text{ whenever } \|x\|_\infty \leq R.$$

The minimal such R is called the range of the interaction.

The connection between the rates of flip and the Hamiltonian (1) and the temperature $T = 1/\beta$ is established by imposing conditions which assure that the Gibbs measures are not only invariant, but also reversible with respect to the dynamics. These conditions, called detailed balance, state that for each $x \in \mathbb{Z}^d$ and $\sigma \in \Omega$,

$$c(x, \sigma) = c(x, \sigma^x) \exp(-\beta \Delta_x H_h(\sigma)), \tag{12}$$

where

$$\Delta_x H_h(\sigma) := \sigma(x) \Big(\sum_{y: \{x,y\} \in \mathbb{B}_\Gamma} \sigma(y) + h \Big),$$

which formally equals $H_h(\sigma^x) - H_h(\sigma)$. We will usually make the dependence on h explicit, by writing $c_h(x, \sigma)$ for the rates. There are many examples of rates which satisfy the conditions of detailed balance (12) and also the other hypotheses, (H1) and (H2). The most common examples found in the literature are:

Example 1: Metropolis Dynamics

$$c_h(x, \sigma) = \exp(-\beta(\Delta_x H_h(\sigma))^+),$$

where $(a)^+ = \max\{a, 0\}$ is the positive part of a.

Example 2: Heat Bath Dynamics

$$c_h(x, \sigma) = \frac{1}{1 + \exp(\beta \Delta_x H_h(\sigma))}.$$

Example 3:

$$c_h(x, \sigma) = \exp(-(\beta/2) \Delta_x H_h(\sigma)).$$

Each one of these rates satisfies also the further conditions below which will be needed for the analysis in this paper to be possible.

(H3) (*Attractiveness and monotonicity in* h) If $\eta(x) \le \zeta(x)$ and $h_1 \le h_2$, then

$$\begin{aligned} c_{h_1}(x, \eta) \le c_{h_2}(x, \zeta) \quad &\text{if } \eta(x) = \zeta(x) = -1, \\ c_{h_1}(x, \eta) \ge c_{h_2}(x, \zeta) \quad &\text{if } \eta(x) = \zeta(x) = +1. \end{aligned}$$

(H4) (*Uniform boundedness of rates*) For each temperature T there is $h(T) > 0$ and $0 < c_{\min}(T) \le c_{\max}(T) < \infty$ such that for all $h \in (-h(T), h(T))$ and $\sigma \in \Omega$

$$c_{\min}(T) \le c_h(0, \sigma) \le c_{\max}(T).$$

Each one of the examples presented also satisfies the continuity conditions

$$\lim_{h \searrow 0} c_h(x, \sigma) = c_0(x, \sigma),$$

for all x and σ. Interestingly enough, this is true for all the rates that satisfy detailed balance with respect to $H_h(\cdot)$ and the hypotheses (H1)–(H4). Even more surprisingly, the stronger result below, which says that the 'effect of h on the rates is essentially of order h', holds. The lower bounds will be important for arguing, at least heuristically, that the large droplets of the plus-phase grow relatively fast in the background of the minus-phase. The precise statements in the proposition below have a somewhat technical flavor, by necessity. It is not true that all the rates depend on h; for instance consider Metropolis dynamics, for which some of the rates stay constant, equal to 1, as h varies on a small interval. Thanks to the hypotheses of translation invariance, (H1), we need only consider $x = 0$. Define

$$\begin{aligned} a(\sigma, h) &:= c_h(0, \sigma), \\ b(\sigma, h) &:= c_h(0, \sigma^0), \end{aligned}$$

and

$$g(h) := \sup_{\sigma} |c_h(0, \sigma) - c_0(0, \sigma)|.$$

Then we have

Proposition 1. *Suppose that the rates $c_h(x, \sigma)$ satisfy detailed balance with respect to $H_h(\cdot)$ and also the hypotheses (H1), (H2), (H3) and (H4). Then for each $T > 0$ and $\sigma \in \Omega$ there are finite and positive constants $C_1(T, \sigma)$ and $C_2(T, \sigma)$, such that for all $h \in (-h(T), h(T))$, the following two statements hold:*

(i) $C_1(T, \sigma)|h| \leq \max\{|b(\sigma, h) - b(\sigma, 0)|, |a(\sigma, h) - a(\sigma, 0)|\} \leq C_2(T, \sigma)|h|$.

(ii) $C_1(T, \sigma)|h| \leq \max\{|b(\sigma, h) - b(\sigma, -h)|, |a(\sigma, h) - a(\sigma, -h)|\} \leq 2C_2(T, \sigma)|h|$.

There are also two further positive and finite constants $C_3(T)$ and $C_4(T)$ such that for all $h \in (-h(T), h(T))$, we have

(iii) $C_3(T)|h| \leq g(h) \leq C_4(T)|h|$.

Proof. (ii) follows from (i) and the monotonicity hypotheses, (H3), while (iii) follows from (i) and the finite range hypotheses, (H2), so all we need is to prove (i).

In proving (i) there is no loss in generality in supposing that $\sigma(0) = +1$, since $a(\sigma, h)$ and $b(\sigma, h)$ play symmetric roles. Using this remark, one can also see that there is also no loss in generality in supposing that $h \geq 0$, since otherwise we can use the symmetry of the model with respect to simultaneous mappings of $h \to -h$ and $\sigma \to -\sigma$, to reduce the problem to this case.

The detailed balance condition (12) states that

$$\frac{b(\sigma, h)}{a(\sigma, h)} = \exp\left(\beta \sum_{y:\|x-y\|_1=1} \sigma(y) + \beta h\right) = \frac{b(\sigma, 0)}{a(\sigma, 0)} \exp(\beta h).$$

Hence

$$\log(b(\sigma, h)/b(\sigma, 0)) + \log(a(\sigma, 0)/a(\sigma, h)) = \beta h.$$

From the hypotheses of monotonicity in h, $a(h)$ decreases with h, while $b(h)$ increases with h, therefore the two logarithms above are positive, and hence must vanish as $h \searrow 0$, implying that $b(\sigma, h) \to b(\sigma, 0)$ and $a(\sigma, h) \to a(\sigma, 0)$. Moreover

$$\tfrac{1}{2}\beta h \leq \max\{\log(b(\sigma, h)/b(\sigma, 0)), \log(a(\sigma, 0)/a(\sigma, h))\} \leq \beta h.$$

Hence for small h (depending on σ and β),

$$\tfrac{1}{4}\beta h \leq \max\{(b(\sigma,h)-b(\sigma,0))/b(\sigma,0)), (a(\sigma,0)-a(\sigma,h))/a(\sigma,h))\} \leq 2\beta h.$$

Using now the hypotheses of boundedness of the rates, (H4), we can conclude that there are positive finite constants $C_1(T,\sigma)$ and $C_2(T,\sigma)$ such that

$$C_1(T,\sigma)h \leq \max\{|b(\sigma,h)-b(\sigma,0)|, |a(\sigma,0)-a(\sigma,h)|\} \leq C_2(T,\sigma)h.$$

□

Throughout this paper we will suppose that we have chosen and kept fixed a set of rates $c_h(x,\sigma)$ which satisfy the detailed balance conditions, (12) and all the hypotheses (H1)–(H4). This spin flip system will be denoted by $(\sigma^{\eta}_{h;t})_{t\geq 0}$, where η is the initial configuration. If this initial configuration is selected at random according to a probability measure ν, then the resulting process is denoted by $(\sigma^{\nu}_{h;t})_{t\geq 0}$. The probability measure on the space of trajectories of the process will be denoted by $\mathbb{P}$, and the corresponding expectation by $\mathbb{E}$. (Later, when we couple various related processes, we will also use the symbols $\mathbb{P}$ and $\mathbb{E}$ to denote probabilities and expectations in some larger probability spaces, but no confusion should arise from this.) The assumption of detailed balance, (12), assures that the Gibbs measures are invariant with respect to the stochastic Ising models. Moreover, from the assumption of attractiveness, (H3), one obtains the following convergence results

$$\sigma^{-}_{h;t} \to \mu_{h,-} \quad \text{and} \quad \sigma^{+}_{h;t} \to \mu_{h,+},$$

weakly, as $t \to \infty$.

We will want to consider, sometimes as a tool, and sometimes for its own sake, the counterpart of the stochastic Ising model that we are considering, on an arbitrary finite set $\Gamma \in \mathcal{F}$, with some boundary condition $\xi \in \Omega$. This process, which will be denoted by $(\sigma^{\eta}_{\Gamma,\xi,h;t})_{t\geq 0}$, where $\eta \in \Omega_{\Gamma,\xi}$ is the initial configuration, is defined as the spin flip system with rates of flip given by

$$c_{\Gamma,\xi,h}(x,\sigma) = \begin{cases} c_h(x,\sigma) & \text{if } \sigma,\sigma^x \in \Omega_{\Gamma,h}, \\ 0 & \text{otherwise.} \end{cases}$$

When $\sigma,\sigma^x \in \Omega_{\Gamma,h,}$, (12) yields, for all $x \in \mathbb{Z}^d$,

$$\mu_{\Gamma,\xi,h}(\sigma)c_{\Gamma,\xi,h}(x,\sigma) = \mu_{\Gamma,\xi,h}(\sigma^x)c_{\Gamma,\xi,h}(x,\sigma^x), \tag{13}$$

which is the usual reversibility condition for finite state-space Markov processes. (Conversely, if one requires (13) to be satisfied for arbitrary $\Gamma \in \mathcal{F}$ and $\xi \in \Omega$, then one can deduce that (12) must hold.) It is clear from (H4) that $(\sigma^{\eta}_{\Gamma,\xi,h;t})$ is irreducible and hence from (13) it follows that, for any η,

$$\sigma^{\eta}_{\Gamma,\xi,h;t} \to \mu_{\Gamma,\xi,h},$$

weakly, as $t \to \infty$.

Graphical Construction: The experts on interacting particle systems can safely skip this part. We present the graphical construction below only for the benefit of readers who may wonder how the processes described above could be constructed. The elementary graphical construction presented below is one of the possible answers, and has the advantage for our purposes of letting one construct all the systems with the different initial configurations, on all the subsets of $\mathbb{Z}^d$, with the different boundary conditions, all on the same probability space. (One refers to such a construction as a 'coupling' of these various processes.) Moreover, two types of intuitively clear features are then implemented in this coupling:

1. The finite range of the interaction causes 'effects to travel with a bounded speed' (see Proposition 2).
2. Attractiveness and monotonicity in h cause the coupling to preserve the order between the marginal processes (see the inequalities (14), (15) and (16)).

The construction below is a specific version of what is called basic coupling between spin flip processes: a coupling in which the spins flip together as much as possible, considering the constraint that they have to flip with certain rates. The construction is carried out by first associating to each site $x \in \mathbb{Z}^d$ two independent Poisson processes, each one with rate $c_{\max}(T)$. We will denote the successive arrival times (after time 0) of these Poisson processes $(\tau^+_{x,n})_{n=1,2,\ldots}$ and $(\tau^-_{x,n})_{n=1,2,\ldots}$. Assume that the Poisson processes associated with different sites are also mutually independent. We say that at each point in space-time of the form $(x, \tau^+_{x,n})$ there is an upward mark and that at each point of the form $(x, \tau^-_{x,n})$ there is a downward mark. Next we associate to each arrival time $\tau^*_{x,n}$, where $*$ stands for $+$ or $-$, a random variable $U^*_{x,n}$ with uniform distribution between 0 and 1. All these random variables are supposed to be independent among themselves and independent from the previously introduced Poisson processes. This finishes the construction of the probability space. The corresponding probability and expectation will be denoted, respectively, by $\mathbb{P}$ and $\mathbb{E}$. We have to say now how the various processes are constructed on this probability space. For finite Γ and arbitrary ξ, the process $(\sigma^\eta_{\Gamma,\xi,h;t})$ is constructed as follows. We know that almost surely the random times $\tau^*_{x,n}$, $x \in \Gamma$, $n = 1, 2, \ldots$, $* = +, -$, are all distinct, and we update the state of the process at each time when there is a mark at some $x \in \Gamma$ according to the following rules. If the mark that we are considering is at the point $(x, \tau^*_{x,n})$, and the configuration immediately before time $\tau^*_{x,n}$ was σ, then

(i) The spins not at x do not change.
(ii) If $\sigma(x) = -1$ (resp. $\sigma(x) = +1$), then the spin at x can only flip if the mark is of upward type (resp. downward type).
(iii) If the mark is upward and $\sigma(x) = -1$, or if the mark is downward and $\sigma(x) = +1$, then we flip the spin at x if and only if $c_{\Gamma,\xi,h}(x, \sigma) > U^*_{x,n} c_{\max}$.

One can readily see that the process constructed in this fashion has the correct rates of flip.

In principle, one would like to construct the processes on the infinite lattice $\mathbb{Z}^d$ in a similar fashion, with $c_h(x, \sigma)$ replacing $c_{\Gamma,\xi,h}(x, \sigma)$ in (iii). Some extra care has to be taken, because during any non-degenerate interval of time infinitely many marks occur. This is not a real problem, because of the assumption that the range of the interaction, R, is finite. Starting from a configuration η at time 0, we have to say how the spin at a generic site x at a time t is obtained. We will argue that on a set of probability 1 in the space where the marks were defined, for any fixed x and

t, if we take any boundary condition ξ, then the sequence $(\sigma^{\eta}_{\Lambda(l),\xi,h;t}(x))_{l=1,2,\ldots}$ will converge as $l \to \infty$ (i.e., will become constant for large l), to a limit which does not depend on ξ. This limit can then be taken to be the value of $\sigma^{\eta}_{h;t}(x)$, and it is clear that the version of the process $(\sigma^{\eta}_{h;t})$ constructed in this fashion has the correct flip rates. To prove the claim above about insensitivity to receding boundary conditions, we introduce the events $E(x,t,l)$ that there exists a sequence of points in space-time $(x_0,0),(x_1,t_1),\ldots,(x_n,t_n)$ with the properties that $0 < t_1 < \ldots < t_n < t$, $x_0 \notin \Lambda(l)$, $x_n = x$, $\|x_i - x_{i-1}\| \le R$ for $i = 1,\ldots,n$, and that at each point (x_i,t_i), $i = 1,\ldots,n$, there is a mark. It is easy to see that out of the event $E(x,t,l)$, $\sigma^{\eta}_{\Lambda(l),\xi,h;t}(x)$ does not depend on ξ. Because $E(x,t,l) \subset E(x,u,l)$, when $t \le u$, our claim is reduced to the statement that for each x and integer t, $E(x,t,l)$ happens for only finitely many values of l, $\mathbb{P}$-almost surely. We have to show now that the probability of $E(x,t,l)$ vanishes fast enough as $l \to \infty$, so that we can apply the Borel–Cantelli Lemma. In order to do it we observe that for a given n, the sites $x_0,\ldots,x_{n-1}$ in the definition of $E(x,t,l)$ cannot be chosen in more than $((2R+1)^d)^n$ ways. Also, for large l, n cannot be less than $l/(3R)$. Therefore,

$$\mathbb{P}(E(x,t,l)) \le \sum_{n \ge l/(3R)} (2R+1)^{dn}\mathbb{P}(Z \ge n),$$

where Z is a Poisson random variable, with mean $tc_{\max} =: r$. We will use now the following elementary inequality, valid for $n \ge r$,

$$\begin{aligned}\mathbb{P}(Z \ge n) \;&=\; e^{-r}r^n \sum_{k \ge n} \frac{r^{k-n}}{k!} \le e^{-r}r^n \sum_{k \ge 0} \frac{n^{k-n}}{k!} \\ &=\; (r/n)^n e^{n-r} \le \exp(-n(\log(n/r) - 1)).\end{aligned}$$

Combining the last two estimates we obtain for large l (depending only on x)

$$\mathbb{P}(E(x,t,l)) \le \sum_{n \ge l/(3R)} (2R+1)^{dn} \exp(-n(\log(l/(3Rtc_{\max})) - 1)),$$

which goes to 0 faster than any exponential of l.

Actually, the estimate above shows also that even if we let t grow with l, but keeping l/t large enough, then the spin at a fixed site x is almost insensitive up to time t to what happens outside of the box $\Lambda(l)$:

Proposition 2. *For each dimension d and temperature T, there exists a finite positive constant $C(d,T)$ such that if we let $l \to \infty$ and $t \to \infty$ together, keeping $l \ge C(d,T)t$, then for every site $x \in \mathbb{Z}$,*

$$\sup_{h \in (-h(T),h(T))} \sup_{\xi \in \Omega} \sup_{\eta \in \Omega} \mathbb{P}(\sigma^{\eta}_{h;t}(x) \ne \sigma^{\eta}_{\Lambda(l),\xi,h;t}(x)) \to 0,$$

exponentially fast in l.

Because of the hypotheses (H3), of attractiveness and monotonicity in h, the coupling provided by the construction above preserves the order between the coupled marginal processes, in various cases. In this paper we will need (particular cases of) the following facts. If $\eta \leq \zeta$, $\xi \leq \xi'$, $-h(T) < h_1 \leq h_2 < h(T)$ and $\Gamma \in \mathcal{F}$ is arbitrary, then for all $t \geq 0$,

$$\sigma^{\eta}_{\Gamma,\xi,h_1;t} \leq \sigma^{\zeta}_{\Gamma,\xi',h_2;t}, \tag{14}$$

$$\sigma^{\eta}_{h_1;t} \leq \sigma^{\zeta}_{h_2;t}, \tag{15}$$

and

$$\sigma^{\eta}_{\Gamma,-,h_1;t} \leq \sigma^{\zeta}_{h_2;t}. \tag{16}$$

We will refer to these inequalities as basic-coupling inequalities. (Observe that the Holley–FKG inequalities for the models we are considering can be derived from (14).)

Two More Remarks on Notation: We will use C, $C(T)$, $C(T,d)$, C_1, C_2, etc., to denote positive finite constants, whose precise values are not relevant and may even change from appearance to appearance.

We will use the notation $\beta' = \beta - \log b$, where b is a technical constant, which depends on the dimension d. These constants b are associated to the notion of 'counting contours'; in $d = 2$, $b = \log 3$. Several times we will encounter the fraction β'/β, and we observe that it satisfies

$$\beta'/\beta = 1 - T\log b \nearrow 1 \quad \text{as} \quad T \searrow 0. \tag{17}$$

3. The Main Result in [Sch3]

The main result in [Sch3] is the following theorem.

Theorem 1. *For each dimension $d \geq 2$ there is $T_0 > 0$ such that for every temperature $T \in (0, T_0)$ the following happens. There are constants $0 < \lambda_1(T) \leq \lambda_2(T) < \infty$ such that if we let $h \searrow 0$ and $t \to \infty$ together, then for every local observable f,*

(i) $\mathbb{E}\left(f(\sigma^{-}_{h;t})\right) \to \int f\,d\mu_-$ *if* $\limsup h^{d-1}\log t < \lambda_1(T)$,

(ii) $\mathbb{E}\left(f(\sigma^{-}_{h;t})\right) \to \int f\,d\mu_+$ *if* $\liminf h^{d-1}\log t > \lambda_2(T)$.

We can take $\lambda_1(T) = (2^d(d-1)^{d-1}/(d+1))(\beta'/\beta)^d\beta$, and $\lambda_2(T) = (2^d d^{d-1})(1 + \delta(T))\beta$, where $\delta(T)$ is a positive-valued function which vanishes as $T \searrow 0$.

In other words, we are stating that the law of the random configuration $\sigma^{-}_{h;t}$ converges weakly to μ_- in case (i) and to μ_+ in case (ii).

Theorem 1, apart from the explicit estimates on $\lambda_1(T)$ and $\lambda_2(T)$, was conjectured by Aizenman and Lebowitz in [AL], where they proved a similar result for certain deterministic cellular automata evolving from initial random configurations selected according to translation invariant product measures. Actually they conjectured the stronger result, which states that also $\lambda_1(T) = \lambda_2(T) =: \lambda_c(T)$. This is a natural further conjecture, but we believe that it will be extremely difficult to prove it, because it is not even clear what the common value of $\lambda_1(T)$ and $\lambda_2(T)$ should be, as we will explain when we present the heuristics behind Theorem 1.

In contrast to Theorem 1, the proposition below, which is much easier to prove, says that for temperatures for which there is no phase transition, the relaxation to equilibrium occurs in a time of order 1 (no scaling with h).

Proposition 3. *For each dimension d and for every temperature T for which $\mu_- = \mu_+ =: \mu_0$, if we let $h \searrow 0$ and $t \to \infty$ together, then for every local observable f*

$$\lim \mathbb{E}\left(f(\sigma^-_{h;t})\right) = \int f d\mu_0.$$

Proving this Proposition may again be a good exercise for newcomers to the field. The proof is actually provided in [Sch3], so that this is an exercise with solution. Hint: recall the hint given to prove (9) and (10).

Theorem 1 is a rigorous counterpart of a pattern found by researchers who analyzed the relaxation of the stochastic Ising models by simulating the dynamics with computers. One of the best known papers in this regard is [BM]. Translating the results in that paper to our setting, one is running the system under a small positive external field, starting from all spins down (and using periodic or free boundary conditions). One is interested in the time evolution of a local observable, say, the value of the spin at the origin. An average is taken over a large number of independent repetitions of the same evolution from time 0 up to a certain time. Under these conditions there is manifestation of metastable behavior in the form of a 'plateau' in the relaxation curve that is obtained: in a relatively short time the average value of the spin at the origin seems to converge to a value close to the opposite of the spontaneous magnetization, after this, one sees an apparent flatness in the relaxation curve over a stretch of time which may be quite long compared with the time needed to first approach this value. But eventually the relaxation curve starts to deviate from this constant value and move upwards, towards the true asymptotic limit, close to the spontaneous magnetization. The experimentally-almost-flat portion of the relaxation curve is referred to as a plateau. Of course, for given values of the parameters T and h the relaxation curve is strictly monotone increasing, and there is no clear cut definition of what the plateau is. On the other hand, repeating the numerical experiment with smaller and smaller values of h (at the same temperature T) one sees that the flatness becomes more and more evident, in the sense that the first portion of the relaxation curve, which is observed while the system is moving towards its 'metastable state' becomes essentially independent of h, while the length of the apparent plateau increases (see Figures 6, 7 and 8 in [BM]).

Theorem 1, besides giving a precise mathematical meaning to the idea that a plateau seems to be approached for a very long time but that eventually the system moves away from that plateau, provides a good estimate of how long the plateau is when h is very small: its length is of the order of an exponential of $(1/h)^{d-1}$, as $h \searrow 0$. This quantitative feature was expected to hold, based on the heuristics which will be presented in the next section; it was also observed in simulations: see [Sta]. In contrast, the following result, which is much easier to prove, already implies the existence of a plateau. The proof works for every $T < T_c$, but it has the important disadvantage of not giving, by any means, a good estimate on the length of the plateau. Its proof is a simple combination of Propositions 1 and 2.

Proposition 4. *For each dimension $d \geq 2$ and every temperature $T \in (0, T_c)$ the following happens. If we let $h \searrow 0$ and $t \to \infty$ together in such a way that* $\limsup ht^{d+1} = 0$*, then for every local observable f*

$$\mathbb{E}\left(f(\sigma^-_{h;t})\right) \to \int f d\mu_-.$$

A second aspect of the relaxation pattern which can be seen in the simulations and which is another reason for much of the interest in the problem is the particularly relevant role played by the behavior of individual droplets of spins $+1$ (possibly with holes where the spins are $-$) in the sea of spins -1, during the evolution. In the 'metastable state' one sees such droplets appearing spontaneously throughout the system, but shrinking and disappearing before they become large, in a sort of equilibrium which resembles the minus-phase. Eventually one of these droplets grows to a larger size, apparently by chance, and then it keeps growing and eventually 'covers' the whole system, which is then in the true equilibrium phase. While this droplet is growing, it sometimes happens that other large droplets appear somewhere else and also grow, so that the system is, in this case, driven to equilibrium when such droplets coalesce and 'cover' the system. This phenomenon, which is also observed in real experiments (see the reviews quoted in the introduction), is known as 'nucleation and growth'. Many theoretical and numerical studies have focused on these aspects of the evolution and on simplified, single-droplet, or independent-droplets, models. It is a common saying that one can 'understand' the behavior of the individual droplets on purely 'energetic', or rather 'free-energetic' terms, as a problem of escaping from a potential well. A very heuristic form of this reasoning will be reviewed in the next section, and indeed served to orientate our approach towards proving rigorous results.

4. Heuristics

We present now the heuristics behind Theorem 1. This heuristic reasoning comes in two parts, the first one of which is very well known, while the second one seems to have escaped most of the attention.

First Part: We want to consider the behavior of an individual droplet of spins $+1$ in a background of spins -1. When the temperature is low, it is reasonable, on energetic grounds, to consider simply a cube full of spins $+1$, the other spins being all -1, as such a droplet. If the side-length of the cube is l, then the energy of such a configuration, with respect to the energy of the configuration with all spins -1 is given by

$$e_h(l) := 2dl^{d-1} - l^d h.$$

As a function of l, considered now as a continuous quantity, $e_h(l)$ grows from 0 to its maximum

$$E_{\max} = 2^d(d-1)^{d-1}/h^{d-1},$$

when l varies from 0 to $l_c = 2(d-1)/h$. For $l > l_c$, $e_h(l)$ decreases; it crosses the value 0 when $l = 2d/h$, and goes to $-\infty$ when $l \to \infty$.

If we assume that the droplet evolves in such a way as to lower the energy of the system, then we are led to the conclusion that droplets with side-length smaller than l_c tend to shrink and that droplets with side-length larger then l_c tend to grow and cover the whole system. Also by analogy with other phenomena related to passage over potential barriers, one would expect that the time needed for a droplet to pop up spontaneously, due to a thermal fluctuation, *in a given place* is of the order of

$$\exp(\beta E_{\max}),$$

which grows exponentially with $1/h^{d-1}$.

Second Part: From the discussion above one could naively predict for the system a relaxation time of the order of $\exp(\beta E_{\max})$. Actually, this is only reasonable if the whole system is not much larger than the size of a critical droplet, so that the time for such a droplet to first appear should indeed be of that order and, moreover, when such a droplet appears, it will cover the whole system in a comparably negligible time. For instance, this seems to be a good prediction if the linear size of the system scales as B/h with a large fixed B. (In this regard, see the next section, Theorem 4 in Section 6, and Corollary 2 in Section 7). But we are concerned with a larger (infinite) system, and we are observing it through a local function f, which depends, say, on the spins in a finite set S. For us the system will have relaxed to equilibrium when S is covered by a big droplet of the plus-phase, which appeared spontaneously somewhere and then grew, as discussed above. We want to estimate how long we have to wait for the probability of such an event to be large. If we suppose that the radius of supercritical droplets grows with a fixed speed v, then we can see that the region in space-time where a droplet which covers S at time t could have appeared is, roughly speaking, a cone with vertex in S and which has as base the set of points which have time-coordinate 0 and are at most at distance tv from S. The volume of such a cone is of the order of $(vt)^d t$. Now, from the discussion in the first part of the heuristics, one can infer that 'the rate with which supercritical droplets appear by thermal fluctuations' at a given location should be of the order of $\exp(-\beta E_{\max})$. The order of magnitude of the relaxation time, t_{rel}, before which the region S is unlikely to have been covered by a large droplet and after which the region S is likely to have been covered by such an object can now be obtained by solving the equation

$$(vt_{\text{rel}})^d t_{\text{rel}} \exp(-\beta E_{\max}) = 1.$$

This gives us

$$t_{\text{rel}} = v^{-d/(d+1)} \ \exp(\beta E_{\max}/(d+1)).$$

In order to use this relation to predict the way in which the relaxation time scales with h, one needs to figure out the way in which v scales with h. If we suppose, for instance, that v does not scale with h, or that at least it goes to 0, as $h \searrow 0$, so slowly that

$$\lim_{h \searrow 0} h^{d-1} \log v = 0, \tag{18}$$

then we can predict that

$$t_{\text{rel}} \sim \exp(\beta E_{\max}/(d+1)) = \exp\left(\frac{\beta 2^d (d-1)^{d-1}}{(d+1)h^{d-1}}\right). \tag{19}$$

We will explain now why it seems reasonable to suppose that (18) is true. v should be the asymptotic speed of growth of the droplet, when it becomes very large (much larger than the critical size), and in this regime we can neglect the curvature of the surface of the droplet and regard the growth of its radius as resulting from the movement of its boundary as that of a (mesoscopically) flat interface, caused by the fact that h is positive. Thinking of the surface as a roughly flat interface and keeping in mind that h is small, we can, in first approximation, assume that on one side of the interface we have the minus-phase and on the other side the plus-phase, which are symmetric, and that protuberances of each phase into the other at the interface are essentially similar. The movement of the interface is then caused simply by the larger rate of flip of spins in the upward direction, caused by the fact that $h > 0$, when we compare two situations which are related by spin reversal at all sites. From part (ii) of Proposition 1, one can see that this difference in the rates of flip caused by the external field h is of the order of h. Therefore one obtains

$$v \sim h \text{ as } h \searrow 0,$$

which implies (18).

From (19) one sees that the relaxation time, even for the infinite system, should grow exponentially with h^{1-d}, and what the rate of this exponential growth should approximately be when T is close to 0. The fact that in part (i) of Theorem 1 we have $\lambda_1(T)$ which is asymptotic as $T \searrow 0$ to the value of $\lambda_c(T)$ predicted in (19) is a pleasant feature of the method used to prove this side of the Theorem. On the other hand, in part (ii) of Theorem 1 we are missing the factor $1/(d+1)$, in $\lambda_2(T)$ because we are not able to control rigorously the growth of the supercritical droplets and make complete sense out of (18). The other factor by which $\lambda_2(T)$ differs from $\lambda_c(T)$ even as $T \to 0$ is there for other technical reasons.

A major question, which seems to be controversial even from a heuristic standpoint, is the prediction of the correct value of $\lambda_c(T)$, for each T (small enough, if necessary), and not just its asymptotic behavior as $T \to 0$. A certain type of 'common wisdom' says that one should repeat the computation above but with the cubes replaced by solids which have the Wulff shape corresponding to the surface tension at temperature T. This idea has, nevertheless, been challenged by the results obtained in the limit of very low temperature, in which h is kept small but fixed and T is scaled to 0, by Kotecký and Olivieri in [KO2] and [KO3] (results announced in [KO1]). (After discussions with these two colleagues, it seems to me that there is no compelling evidence that in the limit considered here, in which T is small but fixed and $h \searrow 0$, Wulff shapes should be more likely to come into play in this problem than in the limit of very low temperatures.) In connection with this discussion, one may want to refer to the fact that investigations have been carried out on simulations and analytic (non-rigorous) studies of supercritical droplet growth (see for instance [DS] and references therein). Nevertheless such investigations refer to the growth of droplets which are very supercritical and should develop an asymptotic shape related to the different asymptotic speed of growth in different directions. The asymptotic shape is not given by the equilibrium Wulff construction, but by a similar construction based on the speed of growth as a function of the direction. In any case this asymptotic shape obtained when a droplet is moving downhill, 'with the drift', does not clarify the controversy about the first droplets which appear and are likely to grow (a completely different, large-deviations type problem, related to moving uphill, 'against the drift').

5. Rigorous Counterparts to the Equilibrium Notion of Critical Droplets

In this section we consider the Gibbs measure $\mu_{\Lambda(B/h),-,h}$. We want to take $B > 2d$, so that, in heuristic terms, we are able to comfortably fit a supercritical droplet with negative energy inside this box. More precisely, the computation in the last section can be used to show that the only configuration that minimizes the energy is then the configuration with all spins $+1$ inside of $\Lambda(B/h)$, and that this is false if $B < 2d$.

Below we will present two basic facts about these finite systems which were proven in [Sch3], and which are rigorous counterparts to what the heuristics tells us. The fact that we are considering $(-\underline{1})$-boundary conditions is crucial to allow one to use the systems we are considering here as building blocks in the analysis of larger systems.

We will denote by $\mathcal{B}$ the set of configurations in Ω in which the box $\Lambda(d/h)$ intersects an infinite cluster of spins -1. Observe that if $\sigma \in \Omega_{\Lambda(B/h),-}$ and $\sigma \notin \mathcal{B}$, then in the configuration σ, $\Lambda(d/h)$ is surrounded by a shell of $+1$ spins which separates it from $\partial_{\text{ext}}\Lambda(B/h)$. For this reason, Theorem 2 below can be seen as a rigorous counterpart to the idea that when the process $(\sigma_{\Lambda(B/h),-,h;t})$ is in equilibrium, a droplet which covers the core of the system is present (once we know that the shell mentioned above is present, one can use the Markov property of the Gibbs measures in a standard fashion, by considering the outermost such shell, and conclude, from the Holley–FKG inequalities, that inside this shell the distribution is even higher than the plus-phase).

Theorem 2. *For each $B > 2d$, there exists $T(B) > 0$ so that for all $T \in (0, T(B))$*

$$\lim_{h \searrow 0} \mu_{\Lambda(B/h),-,h}(\mathcal{B}) = 0.$$

This theorem is a strengthening of the main result in [Mar], where B had to be taken large enough (significantly larger than $2d$) regardless of the temperature. Curiously enough, the proof of Theorem 2 in arbitrary dimension is technically the most difficult part of [Sch3]. In that paper a much simpler proof is also presented in the appendix for the special case $d = 2$.

Using Theorem 2, one can easily prove part (ii) of the next theorem. Part (i) of that theorem was proven in [Sch3], using a type of 'Peierls argument' in the presence of an external field, borrowed from [CCO]. Observe that Theorem 3 is a sort of analogue for equilibrium of what Theorem 1 is for the dynamics; the result is more satisfactory here, since the constants $B_1(T)$ and $B_2(T)$ which appear in Theorem 3 both converge to the same limit, $2d$ (predicted by the heuristics), as $T \searrow 0$.

Theorem 3. *For each dimension $d \geq 2$ there is $T_0 > 0$ such that for every temperature $T \in (0, T_0)$ the following happens. There are constants $0 < B_1(T) \leq B_2(T) < \infty$ such that if we let $h \searrow 0$, then for every local observable f*

(i) $\langle f\rangle_{\Lambda(B/h),-,h} \to \int f d\mu_-$ *if* $B < B_1(T)$.

(ii) $\langle f\rangle_{\Lambda(B/h),-,h} \to \int f d\mu_+$ *if* $B > B_2(T)$.

We can take $B_1(T) = 2d(\beta'/\beta)$, *and* $B_2(T) = 2d(1+\delta(T))$, *where* $\delta(T)$ *is a positive-valued function which vanishes as* $T \searrow 0$.

6. Relaxation Time, Rate of Expenential Convergence and Gap in the Spectrum of the Generator

There are at least three different ways to look at the 'speed of relaxation to equilibrium' of the stochastic Ising models. The relaxation time, or rather the relaxation time as a function of a 'precision' parameter ϵ, which is supposed to be positive and small is defined by

$$t^\epsilon := \inf\{t \geq 0 : \mathbb{E}\,(\sigma_t^+(0)) - \mathbb{E}\,(\sigma_t^-(0)) < \epsilon\}.$$

The facts that in this definition only the observable identical to the value of the spin at the origin appears and only the extreme configurations $+\underline{1}$ and $-\underline{1}$ appear as initial configurations are natural, due to the inequalities (15), and to translation invariance.

The rate of exponential convergence to equilibrium is defined as

$$\begin{aligned}\gamma = \sup\{a \geq 0 :\ & \text{there exists } C < \infty \text{ such that}\\ & \mathbb{E}\,(\sigma_t^+(0)) - \mathbb{E}\,(\sigma_t^-(0)) \leq Ce^{-at} \text{ for all } t \geq 0\}.\end{aligned}$$

Finally the third quantity is the gap in the spectrum of the generator L of the stochastic Ising model, extended to act as an operator on $L^2(\Omega, \mu)$. Because the spectrum of L lies in $(-\infty, 0]$, and 0 is in the spectrum (see, e.g., Section 4 of Chapter IV in [Lig]), the gap is defined as

$$\text{gap} = \inf\{x > 0 : x \in \text{spectrum of } -L\}.$$

When we want to make the dependence on h explicit, we write t_h^ϵ, $\gamma(h)$ and gap(h), and we use also similar notation for the processes on finite subsets of $\mathbb{Z}^d$.

A consequence of Theorem 1 is the following corollary.

Corollary 1. *For each dimension* $d \geq 2$, *let* T_0, $\lambda_1(T)$ *and* $\lambda_2(T)$ *be as in Theorem 1. Suppose* $T < T_0$, *then for all small enough but otherwise arbitrary positive* ϵ, *for every* $\lambda' < \lambda_1(T)$ *and* $\lambda'' > \lambda_2(T)$,

$$\exp(\lambda'/h^{d-1}) < t_h^\epsilon < \exp(\lambda''/h^{d-1}),$$

for all small positive h.

To prove this result, given Theorem 1, all one needs is to control the approach to equilibrium starting from $+\underline{1}$ (which actually occurs in a time which does not scale with h). For this one can use the same arguments used to prove Proposition 3.

In recent years, a great deal of effort has been dedicated to establishing relations between γ and gap, in the context of much more general lattice systems, and to the question of proving that each one of them is positive, for values of the parameters (T and h in our case) which are away from phase coexistence regions. This project is an important and very active field of research, and reviewing it here would be beyond the scope of this text. So we limit ourselves to referring the reader to the most recent papers [SZ], [LY], [MO1], [MO2], and [MOS], for much more on the general problem and references to the earlier literature. For our case, it was proven in [MO1] (Theorem 5.1, part (b)) that there exists $T_0 > 0$ so that for all $T \in (0, T_0)$, and every $h > 0$,

$$\mathrm{gap}(h) \geq \gamma(h) > 0.$$

In [MOS] it was also proven that in case $d = 2$, then under the same conditions

$$\gamma(h) = \mathrm{gap}(h).$$

(The reader should be aware that the restrictions above are believed to be just technical. The same results are expected to hold in every dimension, for arbitrary (T, h), except at the transition line and critical point $(0, T_c] \times \{0\}$.)

The positivity of $\gamma(h)$ and $\mathrm{gap}(h)$ is usually referred to as 'rapid convergence to equilibrium'. The fact that it occurs in the regime where the system relaxes 'slowly' to equilibrium when started from $-\underline{1}$, as indicated in Theorem 1 and Corollary 1, may seem at first sight as a contradiction. On second thought, though, one realizes that there is no conflict between a slow loss of memory from the initial configuration, if this one is far from the typical ones in equilibrium, and an eventual exponential approach to equilibrium, for much later times. But even after this remark, it may be somewhat surprising that I will raise the conjecture that at low temperature, $\gamma(h)$ is relatively large for small h.

Conjecture 1. *For each dimension $d \geq 2$, if $T < T_c$, then there exists a positive constant $C(T)$ such that for all small positive h*

$$\mathrm{gap}(h) = \gamma(h) \geq C(T)h^2.$$

In particular I believe that for all small $\epsilon > 0$,

$$t_h^\epsilon \gg (\mathrm{gap}(h))^{-1}, \tag{20}$$

in the sense that the ratio between these quantities blows up when $h \searrow 0$. It is worth pointing out that as far as I am aware, all the methods developed for proving the positivity of γ and gap in more general settings, when specialized to the situation that we are considering, do not provide a proof of (20).

Before explaining why I am raising the above conjecture, I will present some arguments which actually go in the oposite direction, and may raise the readers interest.

One can *formally* write down the following 'spectral expansion', by reasoning in terms of the space $L^2(\Omega, \mu_h)$,

$$\begin{aligned}\mathbb{E}\left(f(\sigma^-_{h;t})\right) &= \sum_{i=0,\ldots}\left(\int f\psi_i d\mu_h\right)e^{-\kappa_i t}\psi_i(-\underline{1})\\ &= \int f d\mu_h + \sum_{i=1,\ldots}\left(\int f\psi_i d\mu_h\right)e^{-\kappa_i t}\psi_i(-\underline{1}). \end{aligned} \tag{21}$$

Here the spectrum of $-L$ is being treated as if it were discrete, with ψ_i being the eigenfunctions and κ_i the eigenvalues, in increasing order. In particular $\kappa_0 = 0$ and $\kappa_1 = \text{gap}(h)$. One is then tempted to conclude that a criterion to decide whether the system is close to equilibrium at time t is the comparison between t and $\text{gap}(h)^{-1}$, i.e., that the relaxation time should be of the same order as $\text{gap}(h)^{-1}$.

The argument above is, of course, too crude to be taken as a serious objection to Conjecture 1, but I thought that it should be mentioned because there seems to still be a folklore, sometimes associated to this computation, according to which a long relaxation time should be identified with a small gap in the spectrum of the generator. It is important to stress that it is not only in the physics literature that the association between the inverse of the gap and the relaxation time in metastable situations is made. For example, in the early 80's E. B. Davies published a series of papers with sophisticated mathematical results (in the form of one sided bounds of the type: a small spectral gap in a proper sense leads to some sort of metastability) which seem to have been motivated by this folklore. In his words: "It seems also plausible [Dav1] that the partition of the configuration space commonly introduced to study metastability is connected with the approximate degeneracy of the zero eigenvalue of the generator of the Glauber stochastic dynamics" ([Dav2], p. 541).

Next I want to point out that for finite (and not too large) counterparts of the model we are considering, this folklore is actually (in a sense) correct. Moreover the close relation between gap and the relaxation time for these smaller systems played a key role in the proof of Theorem 1. We want to consider the process $(\sigma_{\Lambda(B/h),-,h;t})_{t\geq 0}$, i.e., the stochastic Ising model in the box $\Lambda(B/h)$, with $(-\underline{1})$-boundary condition, where B is a constant that we will suppose to be larger than $2d$, exactly as in the previous section, so that a critical droplet and also supercritical droplets with negative energy fit inside this box. The following theorem did not appear explicitly in [Sch3], but, as we explain below, its proof was contained in that paper.

Theorem 4. *For each dimension $d \geq 2$ there is $T_0 > 0$ such that for every temperature $T \in (0, T_0)$ and every $B > 2d$ there are constants $0 < C_1(T) \leq C_2(T,B) < \infty$ such that the following holds:*

(i) *For all small $h > 0$,*

$$\exp(-C_2(T,B)/h^{d-1}) \leq \text{gap}(\Lambda(B/h),-,h) \leq \exp(-C_1(T)/h^{d-1}).$$

(ii) *For all small $\epsilon > 0$,*

$$\exp(C_1(T)/h^{d-1}) \leq t^{\epsilon}_{\Lambda(B/h),-,h} \leq \exp(C_2(T,B)/h^{d-1}),$$

provided $h > 0$ is small enough.

We can take for $C_1(T)$ *any number smaller than* $2^d(d-1)^{d-1}(\beta'/\beta)^d\beta$ *and for* $C_2(T)$ *any number larger than* $2B^{d-1}\beta$.

(In (ii) one can actually take for $C_2(T,B)$ any number larger than $2(2d)^{d-1}\beta$, but this is not relevant for the present discussion.) The lower bound on the gap in this theorem is a particular case of Theorem 5 in [Sch3]. The main technique used for proving it was borrowed from [DiS], [SJ], [JS] and [Sin]. The upper bound on the gap in this theorem, appeared in [Sch3] in a discussion at the end of Section 4; its proof relies on the estimate (32) in that paper, which was derived using the techniques from [CCO] alluded to in the previous section. As for the relaxation times, the same techniques from [CCO], combined with appropriate couplings, give the lower bound, while, as we will explain below, the upper bound results from the lower bound for the gap and from the fact that *contrary to what we are proposing for the infinite system*, the relaxation time for the finite (and small) systems that we are considering is essentially bounded above by the inverse of the gap in the spectrum of their generator. Next we will see why this is the case.

For finite systems, (21) above is correct, but even then one has to be careful before neglecting the terms in the last sum (the quantity of terms there grows as the volume grows). The following is a rigorous counterpart to (21) (see a derivation in Subsection 3-ii in [Sch3]).

$$\begin{aligned}\left|\mathbb{E}\, f(\sigma^{\eta}_{\Lambda(B/h),-,h;t}) - \int f\, d\mu_{\Lambda(B/h),-,h}\right| \\ \le (\mu_{\Lambda(B/h),-,h}(\eta))^{-1/2}\|f\|_\infty e^{-\mathrm{gap}(\Lambda(B/h),-,h)t}.\end{aligned}$$

The factor $(\mu_{\Lambda(B/h),-,h}(\eta))^{-1/2}$ in the upper bound provided by (22) may be quite large, when h is small, but the following very rough bound will control it for our purposes:

$$\begin{aligned}(\mu_{\Lambda(B/h),-,h}(\eta))^{-1/2} &\le \left(\frac{\exp(-\beta H_{\Lambda(B/h),-,h}(\eta))}{2^{|\Lambda(B/h)|}\ \max_{\zeta\in\Omega_{\Lambda(B/h),-}}\exp(-\beta H_{\Lambda(B/h),-,h}(\zeta))}\right)^{-1/2} \\ &\le e^{C(\beta)|\Lambda(B/h)|} \le e^{C(\beta)B^d/h^d},\end{aligned}$$

uniformly in small $|h|$ and η. It may seem amazing that such a term is actually small compared with the other ones, but this is indeed the case. If in (22) we take $t = \exp(C/h^{d-1})$, with C larger than $C_2(T,B)$, then this term is beaten by the exponential of exponential of $1/h^{d-1}$, which prevails, and drives the right hand side of (22) to 0 as $h \searrow 0$. This concludes our sketch of the proof of Theorem 4.

The analysis of systems in boxes of side-lengths B/h with $(-\underline{1})$-boundary conditions was crucial, in [Sch3], for proving part (ii) of Theorem 1. The idea was simply to use the basic coupling between $(\sigma^-_{h;t})$ and $(\sigma^-_{\Lambda(B/h),-,h;t})$, so that the the first marginal lies always above the second one. Thanks to part (ii) of Theorem 3, all that was needed was to show that $(\sigma^-_{\Lambda(B/h),-,h;t})$ relaxes to equilibrium in a time of the order of an exponential of $(1/h)^{d-1}$, which is part of Theorem 4.

Having tried to convince the reader that the lower bound on $\gamma(h)$ which appears in Conjecture 1 would be an interesting result, in part because it says that the behavior of the infinite system is different from that of the finite and not-too-large ones, we explain now why we believe that it is true. The argument that we present is, of course, not a rigorous proof, and for this reason we are somewhat sketchy even with the parts of the argument which can be polished and made completely rigorous already. It should become clear that the main reason why we do not have a proof is our lack of mathematical control on the way droplets grow. At the end of this section we present another conjecture, in which we tried to capture the essence of what is missing; a proof of that other conjecture would probably lead also to a proof of Conjecture 1.

In developing the argument below, I got insight from previous work on cellular automata including the so called Bootstrap percolation model; I refer to the papers [And], [Mou] and [AMS]. This goes with the tradition: Theorem 1 was conjectured by Aizenman and Lebowitz in [AL], where they studied the same type of cellular automata.

The argument is somewhat more transparent (and 'closer' to a proof) in the case when the rates $c(x, \sigma)$ depend only on the restriction of σ to the set of nearest neighbors of x (as is the case for the three explicit examples presented in Section 2). For this reason we will be considering this strengthened form of (H2) to hold below.

We want to compare the process starting from the configurations $-\underline{1}$ and $+\underline{1}$, and it is natural to couple them using the basic coupling. Then we have

$$\mathbb{E}\,(\sigma^{+}_{h;t}(0)) - \mathbb{E}\,(\sigma^{-}_{h;t}(0)) = P(\sigma^{+}_{h;t}(0) \neq \sigma^{-}_{h;t}(0)).$$

In other words, we want to estimate the probability of seeing a discrepancy between the two marginals at $(0, t)$. But discrepancies cannot be created 'spontaneously', so that a discrepancy at $(0, t)$ must be caused by discrepancies at previous times. Because of the assumption that the interaction is only between nearest neighbors, one must then be able to find a set of points in space-time, which contains $(0, t)$, is covered by discrepancies, is the union of straight line segments in the time direction and which has the property that we can move from $(0, t)$ to any other point in the set without leaving the set and going only backwards in time (keeping the space position fixed), or jumping from one space-time location to another one at the same time an at a neighboring site. We call the maximal such set the 'discrepancy cluster of $(0, t)$'. It is crucial to observe that if there is a discrepancy at $(0, t)$, then its discrepancy cluster has to intersect the hyperplane $\mathbb{Z}^d \times \{0\}$.

It is clear that at the points where there are discrepancies the spin in the marginal process $(\sigma^{-}_{h;t})$ has to be -1. Hence, looking only at this marginal process, we see that the discrepancy cluster has to stay outside from the droplets (whose boundaries are shells of spins $+1$). We are interested in the asymptotic behavior for large times, so we can afford waiting a time which is so long that typically the whole space is already covered by droplets which coalesced in such a way that a possible discrepancy cluster has little room left to move through. More precisely (remember that we want to make a quantitative prediction), we break the time t into two parts, the first of which is an incubation time, $u = u(h)$, during which droplets appear all over the space. We want to be able to argue that the probability that the discrepancy cluster of the site $(0, t)$ reaches the hyperplane $\mathbb{Z}^d \times \{u\}$ decays fast enough with $t - u$. The idea here is to look at the growth of the supercritical droplets, taking over the space still not covered by them, but also sometimes (very infrequently) shrinking and loosing ground, as a behavior that dominates the growth of a *contact process*!

We explain now the meaning of the last italicized term. The process that we have in mind is actually a discrete time contact process. The properties of this process that we will be mentioning and using later are either well known or, at least, can be proven by standard techniques. We refer the reader to Chapter 6 in [Lig] and to Chapters 4 and 11 of [Dur] for detailed studies of the (continuous-time) contact process. The version that we will use is defined on the lattice $\mathbb{Z}^d$ in discrete time, $0, 1, 2, \ldots$. Each site at each unit of time may be in state 0 (vacant) or state 1 (occupied), and the evolution is implemented by the following scheme. Given the configuration at time s, the configuration at time $s+1$ is constructed in two steps. The first one is deterministic and defines a preliminary configuration at time $s+1$; each occupied site at time s remains occupied and each vacant site becomes occupied if and only if it has at least one of its $2d$ nearest neighbors occupied at time s. The configuration at time $s+1$ is now obtained from this preliminary configuration by replacing the 1's with 0's according to a probabilistic procedure; the simplest one would be to remove each 1 independently with a probability ϵ. In our case we will be forced to consider a somewhat more general procedure, in which the decisions at different sites are not independent, but the dependency structure will be of finite range, with the probability of each 1 being removed still being a constant, ϵ say. The following is one of the basic facts about these contact processes: if ϵ is small the system admits a translation invariant probability measure ν with positive density of 1's.

Before continuing with the argument, we stress again that this is not a rigorous proof, but rather a heuristic argument, and that the notion of 'droplet' used below is indeed somewhat vague. Below we are always referring to the process $(\sigma^-_{h;t})$ when considering the stochastic Ising model. The connection between the original stochastic-Ising-model space-time and the new one will be carried out by dividing the original one into blocks which are translates of $\Gamma := \Lambda(B/h) \times [0, Ah^{-2})$, with B and A large enough, and associating each one of these blocks to a point in the new space-time. More specifically, the new space-time point (i, s) is in correspondence with $\Gamma(i, s) := \Gamma + (Ki, Ah^{-2}s)$, where K is the side-length of $\Lambda(B/h)$, which is an integer close to B/h.

We will say that the contact-process site (i, s) is occupied if during the whole corresponding time interval, $[Ah^{-2}s, Ah^{-2}(s+1))$, the corresponding stochastic-Ising-model space region $\Lambda(B/h) + Ki$ is covered by a droplet. One should now see the connection between the behavior that we expect for the droplets and the dynamics of the contact process. If h is small, then we expect the heuristic predictions to be accurate with large probability. These predictions would be that occupied sites are likely to remain occupied at each time step, and that, if A was chosen large enough, then vacant sites will become occupied by the contact mechanism, and ϵ will be small. The last statement is implied by the heuristics since there is a distance of order $1/h$ for the border of the droplet, which is supposedly moving at a speed of order h, to cross in a time Ah^{-2}. The mechanisms are not independent from site to site, but only finite range dependencies occur.

On the contact-process space-time, we will use the term 'cluster of 0's' for sets of space-times points which are all vacant and which form a connected subset of $\mathbb{Z}^{d+1}$. From what we said before the cluster of discrepancies of $(0,t)$ has to be contained in blocks $\Gamma(i,s)$ which are all vacant. Let s_0 be the value of s for which $(0,u)$ is in $\Gamma(0,s)$, and s_1 be the value of s for which $(0,t)$ is in $\Gamma(0,s)$. Define $\mathbf{E}$ as the event that the site $(0,s_1)$ belongs to a cluster of 0's which intersects the hyperplane $\mathbb{Z}^d \times \{s_0\}$. We can write now

$$\mathbb{P}(\sigma^+_{h;t}(0) \neq \sigma^-_{h;t}(0)) \leq \mathbb{P}(\mathbf{E}).$$

The argument will be completed once we argue that we can pick $u = u(h)$ so that $\mathbb{P}(\mathbf{E})$ has to decay exponentially, with $s_1 - s_0$ (which is equal to $\lfloor (t-u)/(Ah^{-2}) \rfloor$) with a rate which is h-independent, since then

$$\mathbb{P}(\sigma^+_{h;t}(0) \neq \sigma^-_{h;t}(0)) \leq C_3 \exp(-C_4(s_1 - s_0)) \leq C(h) \exp((-C_4/A)h^2 t),$$

from which the inequality $\gamma(h) \geq (C_4/A)h^2$ follows.

Now we will explain why the claim above on the decay of $\mathbb{P}(\mathbf{E})$ should be correct. Take $u := \exp(C_5/h^{d-1})$, where the constant C_5 will be taken conveniently large, so that the probability that a fixed site, the origin say, is occupied at time s_0 converges to 1 as $h \searrow 0$. Actually we want this probability to be very large, even if the box $\Lambda(3B/h)$ is separated from the rest of the system by $(-\underline{1})$-boundary conditions. This means that we are willing to make the incubation time so large that supercritical droplets will by then have popped up at most places, spontaneously, even if the region was artificially separated from the rest of the system. Let ρ be the distribution of the contact-process configuration at time s_0. From the last remark, we see that the restriction of ρ to the sublattice $3\mathbb{Z}^d$ lies above a product measure with density $\frac{1}{2}$ if h is small. To estimate $\mathbb{P}(\mathbf{E})$ we will use this fact and recall also that ϵ can be made arbitrarily small by taking h small enough (the contact process can be considered to be in the so called 'Peierls regime'). A solution to our problem can be given now by first comparing the contact process started from ρ at time s_0 to the same contact process started from its invariant measure ν at time $-\infty$. One can show that with a sufficiently large probability, the two processes, when coupled in a standard fashion, become identical in a space-time region, Δ, around $(0,s_1)$ with diameter proportional to $s_1 - s_0$ (see Chapter 11 of [Dur] for related, but more delicate, results). The argument is then completed by bounding the probability that in the stationary process the site $(0,s_1)$ belongs to a cluster of 0's which is not contained in this region Δ (below we call this event $\mathbf{F}$). This can be done with 'contour arguments'. Details are quite standard, but very cumbersome, and so we only sketch them. In order to use the standard contours associated with contact processes, we first take the sublattice $\mathcal{L}$ of the contact-process space time $\mathbb{Z}^{d+1}$ which contains the sites (i,s) with the property that the sum of s with the components of i has the same parity as s_1 (so that $(0,s_1) \in \mathcal{L}$). Contours can be thought of as surfaces which separate vacant from occupied sites in $\mathcal{L}$. It is not hard to see that if the event $\mathbf{F}$ occurs, then there must be a family of contours surrounding the restriction of the vacant cluster of the site $(0,s_1)$ to the sublattice $\mathcal{L}$ and that, moreover, this family of contours must be 'connected' in the sense that one can move continuously from any point on any of these contours to any other such point without leaving this surfaces. For this reason the number of families of contours in the conditions above, with total surface l grows at most as a certain exponential of l. Because of the definition of the event $\mathbf{F}$, the minimum value for this total surface is a given multiple of $s_1 - s_0$. From this point on, the argument is completely standard since the dependency structure in the procedure of erasing 1's is of finite range and ϵ can be made as small as needed. This completes our argument in support of Conjecture 1.

From the discussion above it should be clear that one of the first questions to address is that of the growth of the supercritical droplets when h is small. If we admit that this growth is of the type discussed in the heuristics, similar to the movement of essentially flat interfaces, then we should try to understand this mechanism in situations which are not so complex. A simpler situation in which such a movement of a flat interface should occur is that which is produced if we have a box $\Lambda(B/h)$, with B large enough (so that in equilibrium the plus-phase dominates in the bulk), but with the boundary condition in which the spins are -1 on all the faces of $\Lambda(B/h)$, but one of them, where they are $+1$. In this situation, starting from all spins down, we should see the interface move away from this $+$ face, in a fashion similar to what we expect to happen at the border of a large droplet. From the heuristic prediction of a speed $v \sim h$ for the interface, we should expect:

Conjecture 2. *The system just described should relax to equilibrium in a time of order h^{-2}. The gap in the spectrum of the generator in this case should be of order h^2, as $h \searrow 0$.*

Proving this result (even a lower bound for the gap in the form of some large power of h) would probably be a great step towards controlling the growth of the droplets.

7. Finite Systems

From the perspective of Physics, the motivation behind statements as those in Theorem 1, which refer to infinite systems, is actually the idea that such systems are idealizations of very large, but finite, systems. By very large, here, it should be understood that the system is much larger than any relevant space scale in the problem. From the heuristics, it should be clear that, in the present case then, a very large system should be one with linear size of the order of $\exp(D/h^{d-1})$, where D is a large enough constant (how large depends on T also). In such a case the relevant space-time cones introduced in the heuristics will be fully contained inside the system.

Suppose now that we have a smaller system, the process with $-$ boundary conditions evolving in the box $\Lambda(l)$ of side l, say, where l scales in some fashion with h. If l stays constant as $h \searrow 0$, or even if it grows too slowly as $h \searrow 0$, then even in equilibrium the effect of the boundary conditions do not vanish as $h \searrow 0$. From the heuristics this should be clear, since then critical droplets may be larger than the whole system. On the other hand, if we suppose that l grows fast enough to avoid this problem, then we obtained the theorem below, which generalizes Theorem 1.

Theorem 5. *For each dimension $d \geq 2$ there is $T_0 > 0$ such that for every temperature $T \in (0, T_0)$ and every constant $D \geq 0$ the following happens. There are constants $0 < \lambda_1(T, D) \leq \lambda_2(T, D) < \infty$ and $B(T) < \infty$ such that if we let $h \searrow 0$, $t \to \infty$ and $l \to \infty$ together in such a fashion that* $\liminf hl > B(T)$ *and* $\lim h^{d-1} \log l = D$, *then for every local observable f*

(i) $\mathbb{E}\,(f(\sigma^-_{\Lambda(l),-,h;t})) \to \int f d\mu_-$ *if* $\limsup h^{d-1} \log t < \lambda_1(T, D)$.

(ii) $\mathbb{E}\,(f(\sigma^-_{\Lambda(l),-,h;t})) \to \int f d\mu_+$ *if* $\liminf h^{d-1} \log t > \lambda_2(T, D)$.

We can take

$$\lambda_1(T, D) = \max\{2^d (d-1)^{d-1} (\beta'/\beta)^d \beta - dD, (2^d (d-1)^{d-1}/(d+1))(\beta'/\beta)^d \beta\},$$

$\lambda_2(T, D) = (2^d d^{d-1})(1 + \delta_2(T))\beta$, *and* $B(T) = 2d(1 + \delta_3(T))$, *where for* $i = 2, 3$, $\delta_i(T)$ *are positive-valued functions which vanish as* $T \searrow 0$.

Because the statements in Theorem 5 are somewhat involved, we single out next, as a corollary, the particular case in which $D = 0$ (for instance, l may grow as $(1/h)^a$, where $a > 1$, or as $(2.001)d/h$ if the temperature is low enough). This case is conceptually simpler then the case of infinite systems, covered by Theorem 1, in that the notion of 'growth of droplets' is irrelevant here. In particular there is no factor $1/(d+1)$ in the exponential rate of growth of the relaxation time with $1/h^{d-1}$. The asymptotic behavior of $\lambda_2(T, 0)$ as $T \searrow 0$, unfortunately, is still not what one predicts from the heuristics, nevertheless it is interesting to see that, when the dimension becomes large, the ratio between $\lambda_1(T, 0)$ and $\lambda_2(T, 0)$ stays bounded.

Corollary 2. *For each dimension $d \geq 2$ there is $T_0 > 0$ such that for every temperature $T \in (0, T_0)$ the following happens. There are constants $0 < \lambda_1(T, 0) \leq \lambda_2(T, 0) < \infty$ and $B(T) < \infty$ such that if we let $h \searrow 0$, $t \to \infty$ and $l \to \infty$ together in such a fashion that* $\liminf hl > B(T)$ *and* $\lim h^{d-1} \log l = 0$, *then for every local observable f*

(i) $\mathbb{E}\,(f(\sigma^-_{\Lambda(l),-,h;t})) \to \int f d\mu_-$ *if* $\limsup h^{d-1} \log t < \lambda_1(T, 0)$.

(ii) $\mathbb{E}\,(f(\sigma^-_{\Lambda(l),-,h;t})) \to \int f d\mu_+$ *if* $\liminf h^{d-1} \log t > \lambda_2(T, 0)$.

We can take $\lambda_1(T, 0) = 2^d (d-1)^{d-1} (\beta'/\beta)^d \beta$, $\lambda_2(T, 0) = (2^d d^{d-1})(1 + \delta_2(T))\beta$, *and* $B(T) = 2d(1 + \delta_3(T))$, *where for* $i = 2, 3$, $\delta_i(T)$ *are positive-valued functions which vanish as* $T \searrow 0$.

The various constants $\lambda_1(T)$, $\lambda_2(T)$, $\lambda_1(T, D)$, and $\lambda_2(T, D)$, which appear in Theorems 1 and 5 are the best ones that we could obtain rigorously. What should one predict for these constants based on the heuristics? If we suppose that critical droplets are created at a rate $\exp(-\phi(T)\beta/h^{d-1})$, where $\phi(T)$ is a sort of free-energy barrier, that replaces $E_{\max}$, to account for thermal effects, then we should predict for $\lambda_1(T)$ and $\lambda_2(T)$ the common value $\beta\phi(T)/(d+1)$. As we said when the heuristics was presented, it is not clear how $\phi(T)$ should be computed, but on the other hand it seems clear that we should expect that as $T \searrow 0$, $\phi(T) \to E_{\max}$, so that we expect:

Conjecture 3. *Theorem 1 holds with* $\lambda_1(T) = \lambda_2(T) := \lambda_c(T)$, *and*

$$\lim_{\beta\to\infty} \frac{\lambda_c(T)}{\beta} = \frac{2^d(d-1)^{d-1}}{d+1}.$$

How about the finite systems with side-length growing as $h \searrow 0$? Suppose that the side-length grows as $\exp(D/h^{d-1})$, then the heuristics actually predicts the existence of a critical value for D, which separates two regimes. If D is small, then when a supercritical droplet first appears, it grows and covers the whole system in a time that is relatively short. On the opposite case, the droplet takes so long to cover the system, that while it is still growing, other supercritical droplets pop up spontaneously in other parts of the system. The relaxation occurs very much like in the infinite system, with many droplets coalescing. The critical value for D can be easily predicted. Heuristically, the time t_1 for a first critical droplet to appear in the system can be computed via

$$t_1(e^{D/h^{d-1}})^d e^{-\phi\beta/h^{d-1}} = 1,$$

from which we obtain

$$t_1 = e^{(\phi\beta - dD)/h^{d-1}}.$$

On the other hand the time for a supercritical droplet to cover the system after if first appeared is of the order of

$$t_2 = e^{D/h^{d-1}}/v(h) \sim he^{D/h^{d-1}}.$$

Equating t_1 and t_2 we obtain the critical value for D:

$$D_c(T) = \frac{\phi(T)\beta}{d+1}.$$

We state now the corresponding conjecture in a compact form. Observe first that from the discussion before Conjecture 3, we should have $\phi(T)\beta = (d+1)\lambda_c(T)$.

Conjecture 4. *Theorem 5 holds with*

$$\lambda_1(T, D) = \lambda_2(T, D) = \lambda_c(T, D) = \max\{(d+1)\lambda_c(T) - dD, \lambda_c(T)\}.$$

Comparing the numerical values which appear in Theorems 1 and 5 with the corresponding conjectures above, one sees that the lower bounds are much better than the upper bounds. This is so in part because of our lack of rigorous control on the way the droplets grow, as discussed in the previous section.

We turn our attention now to a very interesting idea on how to look at the metastable behavior of systems: the so called 'pathwise approach to metastability', introduced in [CGOV]. Roughly speaking, if one looks not at an average over many realizations of the evolution, as we have been doing here when we take $\mathbb{E}(\cdot)$, but at a single evolution, then one should see a very sharp transition between the metastable and the stable situations. The time taken to make the transition is very short compared with the time spent in the metastable situation, so that essentially the jump is 'instantaneous'. The moment of the jump is nevertheless random and for this reason, when one considers an average over many realization, one sees a much smoother evolution. It is very interesting to see that the distinction between pathwise behavior and average behavior was also realized, apparently independently, by investigators performing simulations. In the paper [TM] the authors emphasize this distinction and observe experimentally sharp pathwise jumps (I am grateful to R. Kotecký for telling me about this paper; see also other references quoted in [TM]).

A detailed pathwise study of the behavior of stochastic Ising models in the limit in which $h \searrow 0$ remains a challenging open problem. From the heuristics one can actually predict two different behaviors for typical paths, depending on how the size of the system grows as $h \searrow 0$. If the side-length l grows 'slowly', in the sense that $\limsup_{h\searrow 0} h^{d-1} \log l < D_c(T)$, then, when a supercritical droplet is first formed, it should cover the system in a relatively short time. In this case we should see a sharp discontinuity in the state of the system for typical path, if we rescale time. This should be so even if we are observing the whole system, for instance through the (non-local) observable $M_t := \sum_{x\in\Lambda(l)} \sigma^-_{\Lambda(l),-,h;t}(x)$. But if l grows 'fast', in the sense that $\liminf_{h\searrow 0} h^{d-1} \log l > D_c(T)$, then we are in the regime in which the whole system relaxes due to many droplets being formed and coalescing, and the evolution of observables like the M_t defined above should be much smoother. In contrast, local observables should still display a jump in their pathwise behavior, reflecting the moment when they are first covered by a supercritical droplet. The only difference in this case, with regard to the smaller systems should be that the rescaled time of the jump should have a distribution that while not degenerate into a constant, should neither be an exponential. This follows from the consideration of the regions in space-time (the cones considered before) where droplets have to be formed, to cover a certain site at a certain time. (But the analysis is actually complicated by the interaction between droplets, when they touch each other.)

One interesting object to look at is the moment when the processes $(\sigma^-_{\Lambda(l),-,h;t})$ and $(\sigma^+_{\Lambda(l),-,h;t})$ 'couple'. By this we mean that we construct both processes on the same probability space, as explained before (basic coupling) and define

$$S := \inf\{t \geq 0 : \sigma^-_{\Lambda(l),-,h;t} = \sigma^+_{\Lambda(l),-,h;t}\}.$$

From the discussion above we conjecture that

Conjecture 5. *Let T and $B(T)$ be as in Theorem 5. Suppose that we let $h \searrow 0$ and $l \to \infty$ together in such a fashion that $\liminf hl > B(T)$ and $\lim h^{d-1} \log l = D$. Then*

$$\frac{S}{\mathbb{E}\,(S)} \text{ converges in distribution to a unit-mean exponential law,}$$

if and only if $D < D_c(T)$.

8. Different Asymptotic Regimes

The results that were proved in [Sch3] and the conjectures discussed above, are always in the form of asymptotics for positive h, when this external field vanishes.

This idea that metastability phenomena should be mathematically described by considering families of processes, indexed by a parameter, and scaling the parameter to zero is not at all new. For fixed values of the parameters h and T, the stochastic Ising models do not seem to display any clear cut, sharp, metastable behavior, but in certain limits, as the one considered here, the behavior of the system becomes closer and closer to what one identifies experimentally as metastable behavior. To some extent this is akin to many other situations in mathematical physics, in which one proves results in the form of limits, with the motivation of understanding the behavior of the system when the scaled parameter is actually fixed (but small or large enough, depending on the case). The thermodynamic limit is certainly an example which comes to mind.

We will refer below to the type of limit considered so far in this paper (T small fixed, $h \searrow 0$) as limit type (i). It is interesting to compare the picture in this limit with the one in the regime in which $h > 0$ is small but fixed and T is scaled to 0. A few years ago, in collaboration with E. J. Neves, the author introduced, in [NS1], [NS2] and [Sch2], an approach which gave a precise mathematical meaning to the notion of critical droplets and metastability for the same stochastic Ising models considered in the present paper, but in this different regime. For a review of this project, in the stage it was in mid 1990, see [Sch1], where also references are given to papers which motivated the approach. Martinelli, Olivieri and Scoppola exploited the results on droplets in this regime to prove rapid convergence to equilibrium in [MOSco1], a topic to which we will return below. The same authors also analyzed similar questions for the Swendsen–Wang dynamics by means of similar analysis of individual droplets in [MOSco2] and [MOSco3]. More recently, further results on these lines appeared in the work of Kotecký and Olivieri, [KO1], [KO2], [KO3], who considered the same type of time evolution that we are considering, but for different Hamiltonians, obtaining interesting differences between the correct patterns of relaxation and some 'common wisdom', at least in this regime. Still more recently, Scoppola, [Sco], presented a general approach to problems of this type, based on the separation of the relevant time scales.

The approach addressed to in the previous paragraph, to which we refer as 'the limit of very low temperatures', or limit of type (ii), is very helpful in clarifying the way in which droplets behave, and how this affects the evolution of the systems. There are various ways in which one can look at this approach. One way to see it is as a laboratory for obtaining insight on what should happen in the more challenging and also, from the point of view of physics, more relevant limit of type (i). Another way to look at this program is as a very valuable project in itself; after all the metastable behavior that is observed for the system in simulations in which T and h are both fixed and different from 0, may be a reflex of the asymptotic behavior in the limit of type (ii) as well as that of type (i). The relaxation patterns of stochastic Ising models are currently much better understood, at a mathematically rigorous level, in this limit of very low temperatures, as compared to the limit of type (i). In two dimensions, these results include an understanding of the mechanism by which supercritical droplets grow, and for this reason provide results on the metastable behavior of finite systems which are quite sharp. Because the size of critical droplets scales with h, but not with T, it makes sense in the case of limit (ii) to consider the system in a box $\Lambda(l)$ with fixed l (large compared with $2/h$), and in this case the metastable behavior and its decay were analyzed in great detail, including the pathwise description of the evolution, as discussed in the previous section.

Results for the infinite system, of the type of those obtained for limit (i) in [Sch3] can also be obtained in the case of limit (ii), and in 2 dimensions, for Metropolis and Heat Bath dynamics, they are sharper than those. In this case one wants to look at the system at time $t = \exp(C\beta)$, for different values of C. One can indeed show that for small values of C (depending on h) one sees locally all spins down and for large values of C, all spins up. Efforts to identify a single critical value $C(h)$, separating the two regimes (i.e., addressing the analogue of Conjecture 3) have failed so far, because it is hard to control the way the speed of growth of the supercritical droplets behaves asymptotically, when the droplets become very large. (The analogue of (18) seems to fail here and one has to find the value of $\lim_{\beta\to\infty}(1/\beta)\log v$.)

The good news is that in spite of the problem just pointed out, we have enough control on the way the droplets grow to obtain, for instance, a result which is analogous to Theorem 1, but in which the denominator $d+1=3$ appears in both, upper and lower bound for $C(h)$. To state the precise result, we need to first recall some facts from [NS1]. Observe that the critical droplet will be an object of size of order 1, since h is fixed. When $2/h$ is an integer some special things happen, and to avoid them we suppose for the moment that this is not the case (in any case, once we have results like Theorems 6 and 7 below for $2/h$ not integer, the same results follows in general by interpolation). We will use the notation $l_c := \lceil 2/h \rceil$ for the smallest integer larger than $2/h$. The critical droplet here is not exactly a square, but rather an object with the following shape: it is a rectangle of sides l_c and $l_c - 1$ plus a single site adjacent to one of the larger sides of this rectangle (so that the smallest rectangle that contains this object is a square of side l_c). The quantity $E_{\max}$ from the heuristics is replaced by the energy associated to the critical droplet just described, and is therefore

$$\Gamma(h) = 4l_c - h((l_c(l_c - 1) + 1),$$

which also has the property $\Gamma(h) \sim 4/h$ as $h \searrow 0$. The best analogue of Theorem 1 and Corollary 1 that we could prove so far (this was done in collaboration with Eduardo Jordão Neves, some time ago) is, with a self-explanatory notation,

Theorem 6. *For Metropolis and Heat Bath dynamics, in $d = 2$, if h is positive and small, there are constants $0 < C_1(h) \le C_2(h) < \infty$ such that if we let $\beta \to \infty$ and $t \to \infty$ together, then for every local observable f*

(i) $\mathbb{E}\left(f(\sigma^-_{\beta;t})\right) \to f(-\underline{1})$ *if* $\limsup(\beta)^{-1}\log t < C_1(h)$,

(ii) $\mathbb{E}\left(f(\sigma^-_{\beta;t})\right) \to f(+\underline{1})$ *if* $\liminf(\beta)^{-1}\log t > C_2(h)$.

We can take $C_1(h) = \Gamma(h)/3$, and $C_2(h) = \Gamma(h)/3 + (2-h)/3$.

Moreover, for every small $\epsilon > 0$, if $C_1' < C_1(h)$ and $C_2' > C_2(h)$, then

$$\exp(\beta C_1') < t_\beta^\epsilon < \exp(\beta C_2'),$$

for all large β.

Observe that we obtain

$$\lim_{h \searrow 0} hC_1(h) = \lim_{h \searrow 0} hC_2(h) = \tfrac{4}{3}.$$

In particular, the difference between $C_1(h)$ and $C_2(h)$ becomes much smaller than each one of these constants, when h is small.

Next we address the analogue of Conjecture 1 in this regime. Because in this regime the notion of a droplet of the plus-phase is essentially that of a region fully covered by $+1$ spins (if the region is not too large, say, not being scaled with β), and because we have in this regime a much better control on the way the droplets grow, we can actually prove the following result.

Theorem 7. *In dimension $d = 2$, for Metropolis and Heat Bath dynamics, if h is small and positive, then for all $\epsilon > 0$*

$$\text{gap}(\beta) = \gamma(\beta) \geq \exp(-\beta(2 - h + \epsilon)),$$

for all large enough β.

The equality in the display was proven in [MO2], and the inequality can be proven using the type of arguments presented in Section 6, in support of Conjecture 1. For this purpose one should take the time length of the blocks used there (which there was Ah^{-2}) as $\exp(\beta(2-h+\epsilon))$ and the side-length K of the same blocks in the space direction as an integer larger than $2d/h = 4/h$. Say that the contact process site i is occupied at time s if, for the stochastic Ising model, in the process $(\sigma^-_{\beta;t})$ at time $s \exp(\beta(2 - h + \epsilon))$ the cube $\Lambda(3K) + Ki$ is fully covered by $+1$ spins and between this time and time $(s + 1) \exp(\beta(2 - h + \epsilon))$ the cube $\Lambda(K) + Ki$ is always fully covered by $+1$ spins. The argument works now because, from the arguments used to prove Theorem 1 in [NS1], we know that during a time of order $\exp(\beta(2-h+\epsilon))$ a square droplet of linear size $3K$, with $K > l_c$ is likely to have grown to cover a concentric square of linear size $5K$, without ever having lost any spin $+1$ in the concentric square of linear size K.

Combining the two theorems above, we see that the analogue of (20) is true here: for small $\epsilon > 0$

$$t^{\epsilon}_{\beta} \gg (\text{gap}(\beta))^{-1}, \tag{22}$$

in the sense that the ratio between these quantities blows up when $\beta \to \infty$.

As far as I know, from the methods which where previously used to prove that $\gamma(\beta) > 0$ for large β as above, the result (22) could not be derived. Curiously enough, the first proof that $\gamma(\beta) > 0$ in this regime was given by [MOSco1], who used also results from [NS1], to obtain a lower bound for $\gamma(\beta)$ of the order of $\exp(-\beta\Gamma(h))$.

While Theorem 7 is interesting because it implies (22), it is probably not the last word. In fact I believe that the gap here should not even vanish as $\beta \to \infty$! More precisely, I propose:

Conjecture 6. *For every dimension d, for Metropolis and Heat Bath dynamics, if h is small and positive, then there exists a constant $C(d, h) > 0$ such that*

$$\text{gap}(\beta) = \gamma(\beta) \geq C(d, h),$$

for all large enough β.

The reason for this conjecture is the fact that when the space is eventually mostly covered by droplets of spins $+1$, these droplets grow not only by the mechanism responsible for Theorem 1 in [NS1] (the appearance of protuberances at the surfaces of flat droplets, at a rate of order $\exp(-\beta(2-h))$), but also by the interaction between droplets which overlap. The point is simply that if a site where the spin is -1 is neighbor to d distinct droplets of spin $+1$, then this spin flips with rate of order 1 (one can call it a 'bootstrap percolation mechanism'). In $d = 2$ for instance, this type of interaction causes overlapping finite rectangular droplets to grow in a time of order 1 until the smallest rectangle that contains both of them is covered with $+1$ spins. In the blocking argument used to prove Theorem 7 we want to take this mechanism into account. We explain next how we plan to do it, but observe that we are still short of a proof. Let the time-length of the blocks $\Gamma(i, s)$ be simply a large constant A, and declare the contact process site i to be occupied at time s if, for the stochastic Ising model, in the process $(\sigma^-_{\beta;t})$ the space-time block $\Gamma(i, s)$ is fully occupied by spins $+1$. Then, instead of seeing a contact process dynamics, we should see a Toom-model type dynamics (as a comparison process, and with some local dependency). By this I mean a model which is similar to the discrete time contact process that was described in Section 6, but in which the preliminary configuration at time $s + 1$ (before the random erasing of 1's) is obtained from the configuration at time s via a bootstrap-percolation-type rule: 1's do not change and a 0 becomes a 1 if and only if it has in each one of the d coordinate directions at least one neighboring 1. The reader is referred to [BG] for an interesting treatment of such systems among others and for references to the former literature. Such systems are known to survive when the probability ϵ of removing 1's is small (because the so called 'eroder condition' is satisfied). And by taking A large and then β large, we can make ϵ as small as we want. To prove Conjecture 6 above one would have to verify that something like the exponential estimates for the contact process used before hold for these processes.

There is actually one case in which we can prove that Conjecture 6 holds: the case $d = 1$! While the limit of type (i) is uninteresting in this case, since $T_c = 0$, the limit of type (ii) leads to metastable behavior, similar to the one that occurs in 2 dimensions. The situation, of course, is much simpler because the critical droplets have size 1, as one can easily check. But this simplicity may be also seen as an advantage, since it makes the one-dimensional system an excellent laboratory to test what we expect to happen in higher dimensions and to get insight. The energy associated to a critical droplet is $2 - h$, and for the relaxation times one obtains precisely the result predicted from the heuristics: for all small $\epsilon > 0$ if $C_1 < (2-h)/2$ and $C_2 > (2 - h)/2$,

$$\exp(C_1\beta) < t^\epsilon_\beta < \exp(C_2\beta),$$

for all large β. Concerning the argument just given to explain why we raised Conjecture 6, we observe that in $d = 1$ the corresponding Toom type model is actually the same discrete-time contact process that was described before, in Section 6. The same contour arguments mentioned there can be used to prove the conjecture in this case. In this case we definitely see that in spite of the metastable behavior of the infinite system, the gap is not vanishing!

Something very interesting happens when one considers a third type of limit.

Once one accepts as natural to scale $h \searrow 0$ or $T \searrow 0$, it becomes also reasonable to ask what happens if we let both vanish together. It turned out that in this regime, we obtained some results which are sharper than in the two other cases, in part because we could use techniques from both cases. The way in which T and h vanish is relevant here, and the analysis is easier if we let $T \searrow 0$ much faster than $h \searrow 0$. On the other hand the case in which we keep a constant ratio between h and T is particularly relevant, because, via a simple transformation, this is equivalent to keeping the temperature and external fields constant, while scaling only the coupling between spins to ∞. We call this type of limit, in which $h \searrow 0$ and $T \searrow 0$, while h/T stays constant, limit of type (iii). In this regime the analogues of Conjectures 3 and 4 are fully vindicated:

Theorem 8. *For Metropolis and Heat Bath dynamics, in $d = 2$, if we let $\beta \to \infty$, $h \searrow 0$ and $t \to \infty$ together, in such a way that βh stays constant (positive) then for every local observable f*

(i) $\mathbb{E}\,(f(\sigma^-_{h,\beta;t})) \to f(-\underline{1})$ *if* $\limsup(h/\beta)\log t < \frac{4}{3}$.

(ii) $\mathbb{E}\,(f(\sigma^-_{h,\beta;t})) \to f(+\underline{1})$ *if* $\liminf(h/\beta)\log t > \frac{4}{3}$.

Moreover, for every small $\epsilon > 0$, if $C_1 < \frac{4}{3}$ and $C_2 > \frac{4}{3}$, then for $h = C/\beta$, where C is an arbitrary positive constant,

$$\exp(\beta C_1) < t^{\epsilon}_{h,\beta} < \exp(\beta C_2'),$$

for all large β.

Theorem 9. *For Metropolis and Heat Bath dynamics, in $d = 2$, for every constant $\delta \geq 0$ the following happens. If we let $\beta \to \infty$, $h \searrow 0$, $t \to \infty$ and $l \to \infty$ together in such a fashion that βh stays constant (positive),* $\liminf hl > 4$ *and* $\lim(h/\beta)\log l = \delta$, *then for every local observable f*

(i) $\mathbb{E}\,(f(\sigma^-_{\Lambda(l),-,h,\beta;t})) \to f(-\underline{1})$ *if* $\limsup(h/\beta)\log t < \max\{4 - 2\delta, \frac{4}{3}\}$.

(ii) $\mathbb{E}\,(f(\sigma^-_{\Lambda(l),-,h,\beta;t})) \to f(+\underline{1})$ *if* $\liminf(h/\beta)\log t > \max\{4 - 2\delta, \frac{4}{3}\}$.

Parts (i) of each one of these theorems follows from techniques used in [Sch3] to prove part (i) of Theorem 1, while parts (ii) rely on careful study of the behavior of individual droplets, in the spirit of the analysis carried out for limit (ii). This analysis of the growth of droplets is more delicate in regime (iii) than in regime (ii), because the corrosion at the four corners of droplets of linear size of the order of B/h is now of order 1, while before it was of order $\exp(-\beta h)$, which was very small. The main reason the results are sharper here is that in spite of our ignorance about the precise speed v with which the radius of supercritical droplets grows, we have effective upper and lower bounds for v that are of the form $\exp(-C\beta)$, with $C \geq 0$ independent of h . In comparison, we are dealing with times of the order of $\exp(C\beta/h)$, for critical droplets to form, so that in the proper time scale, the supercritical droplets grow quite fast, and our lack of knowledge is of secondary order.

From the results and conjectures presented in this paper, it should be clear that one should try to explore the relaxation patterns of stochastic Ising models parametrized by three quantities: h, T and the side-length l of the box in which the system is contained (including the case $l = \infty$). From the heuristics it is usually possible to predict the correct behavior in different regimes, but there is still a substantial distance between most of these heuristic results and their rigorous counterparts. The issues related to relaxation of stochastic Ising models close to the phase-coexistence region, will certainly be the object of mathematical study for many years to come.

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