Math 3C Homework 6 Solutions

Ilhwan Jo and Akemi Kashiwada

ilhwanjo@math.ucla.edu, akashiwada@ucla.edu

Assignment: Section 12.4 Problems 18, 29, 31, 32, 33, 34, 36, 38, 40

18. Suppose that the probability mass functions of a discrete random variable X is given by the following table. Find the mean, the variance, and the standard deviation of X.

x	P(X=x)
-1	0.1
-0.5	0.2
0.1	0.1
0.5	0.25
1	0.35

Solution

$$EX = \sum_{x} xP(X = x)$$

= (-1)(0.1) + (-0.5)(0.2) + (0.1)(0.1) + (0.5)(0.25) + (1)(0.35)
= 0.2850,

$$\begin{split} EX^2 &= \sum_x x^2 P(X=x) \\ &= (-1)^2 (0.1) + (-0.5)^2 (0.2) + (0.1)^2 (0.1) + (0.5)^2 (0.25) + (1)^2 (0.35) \\ &= 0.5635, \\ &\operatorname{var}(X) = EX^2 - (EX)^2 = 0.5635 - 0.2850^2 \approx 0.4823, \end{split}$$

 $\sigma = \sqrt{\operatorname{var}(X)} = \sqrt{0.4823} \approx 0.6945.$

29. Toss a fair coin ten times. Let X be the number of heads.

- (a) Find P(X = 5).
- (b) Find $P(X \ge 8)$.
- (c) Find $P(X \leq 9)$.

Solution

(a) X is binomially distributed with probability $p = \frac{1}{2}$ of having a head. The number of trials is n = 10.

$$P(X=5) = {\binom{10}{5}} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5 = {\binom{10}{5}} \left(\frac{1}{2}\right)^{10}.$$

(b)

$$P(X \ge 8) = P(X = 8) + P(X = 9) + P(X = 10)$$

= $\binom{10}{8} \left(\frac{1}{2}\right)^{10} + \binom{10}{9} \left(\frac{1}{2}\right)^{10} + \binom{10}{10} \left(\frac{1}{2}\right)^{10}$
= $\left(\binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right) \left(\frac{1}{2}\right)^{10} = \left(\frac{10 \cdot 9}{2} + 10 + 1\right) \left(\frac{1}{2}\right)^{10} = \frac{56}{2^{10}}$

(c)

$$P(X \le 9) = 1 - P(X = 10)$$
$$= 1 - {\binom{10}{10}} \left(\frac{1}{2}\right)^{10} = 1 - \frac{1}{2^{10}}$$

31. Roll a fair die six times. Let X be the number of times you roll a 6. Find the probability mass function.**Solution**

Since you roll six times, $X = k = 0, 1, \dots, 6$. Then

$$P(X=k) = {\binom{6}{k}} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{6-k}, \quad k = 0, 1, \cdots, 6.$$

32. An urn contains four green and six blue balls. You draw a ball at random, note its color, and replace it. You repeat this four times. Let X denote the total number of green balls you obtain. Find the probability mass function of X.

Solution

Since you replace each time, each trial is Bernoulli and X is binomially distributed. There are 4 green balls and 6 blue balls. So each time the probability of choosing a green ball is $p = \frac{2}{5}$. There are 4 green balls and you draw 4 times, so X = k = 0, 1, 2, 3, 4.

$$P(X = k) = {4 \choose k} \left(\frac{2}{5}\right)^k \left(\frac{3}{5}\right)^{6-k}, \quad k = 0, 1, \cdots, 4.$$

33. Assume that 20 % of all plants in a field are infested with aphids. Suppose that you pick twenty plants at random. What is the probability that none of them carried aphids?

Solution

We assume that there are sufficiently many plants and each time the probability of picking a plant infested with aphids is p = 0.2. Then

$$P(\text{none is infested with aphids}) = {\binom{20}{0}} (0.2)^0 (1 - 0.2)^{20} = 0.8^{20}$$

34. To test for a disease that has a prevalence of 1 in 100 in a population, blood samples of 10 individuals are pooled and the pooled blood is then tested. What is the probability that the test result is negative (the disease is not present in the pooled blood sample)?

Solution

Let X be the number of blood samples that had the disease. We are told that the probability one person has the disease is $p = \frac{1}{100}$. Since we are testing n = 10 people then

$$P(X=0) = {\binom{10}{0}} \left(\frac{1}{100}\right)^0 \left(1 - \frac{1}{100}\right)^{10} \approx 0.904.$$

36. Toss a fair coin ten times. Let X denote the number of heads. What is the probability that X is within one standard deviation of its mean?

Solution

Since the probability of getting a head on one coin toss is p = 0.5 and we toss the coin n = 10 times, we get

$$\begin{split} \mu &= EX = np = 10(0.5) = 5\\ \sigma &= \sqrt{\operatorname{var}(X)} = \sqrt{np(1-p)} = \sqrt{10(0.5)(1-0.5)} = 1.58.. \end{split}$$

So we want to find $P(\mu - \sigma \le X \le \mu + \sigma) = P(5 - 1.58... \le X \le 5 + 1.58...) = P(3.41... \le X \le 6.58...)$. But we know that the probability only changes when X equals an integer so we are looking for

$$P(X = 4, 5, 6) = {\binom{10}{4}} (0.5)^4 (1 - 0.5)^6 + {\binom{10}{5}} (0.5)^5 (1 - 0.5)^5 + {\binom{10}{6}} (0.5)^6 (1 - 0.5)^4 \\\approx 0.6563.$$

38. A TRUE-FALSE exam has 20 questions. Find the expected number of correct answers if a student guesses the answers at random.

Solution

Let X count the number of answers that are correct. The probability of getting one answer correct is p = 0.5 and there are n = 20 questions so

$$EX = np = 20(0.5) = 10.$$

40. (Sampling with and without replacement) An urn contains K green and N - K blue balls.

(a) You take n balls out of the urn. Find the probability that k of the n balls are green.

(b) You take a ball out of the urn, note its color, and replace it. You repeat this n times. Find the probability that k of the n balls are green.

Solution

Let X be the number of green balls we choose.

(a) We have a total of K + N - K = N balls so there are a total of $\binom{N}{n}$ ways of picking n balls. There are $\binom{K}{k}$ ways of picking k green balls and $\binom{N-K}{n-k}$ ways of choosing n-k blue balls. So

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}.$$

(b) Since we are sampling with replacement, the probability of getting a green ball when choosing one ball is $\frac{K}{N}$. We are choosing n balls so

$$P(X = k) = \binom{n}{k} \left(\frac{K}{N}\right)^k \left(\frac{N-K}{N}\right)^{n-k}.$$