MATH 3B (Winter 2002)
Instructor: Roberto Schonmann
Final Exam

Last Name: Solutions

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Circle the discussion section in which you are enrolled:

2A (Tue. 9am, Madeleine) 2B (Thur. 9am, Madeleine)

2C (Tue. 9am, Brian) 2D (Thur. 9am, Brian)

2E (Tue. 9am, Suneel) 2F (Thur. 9am, Suneel)

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

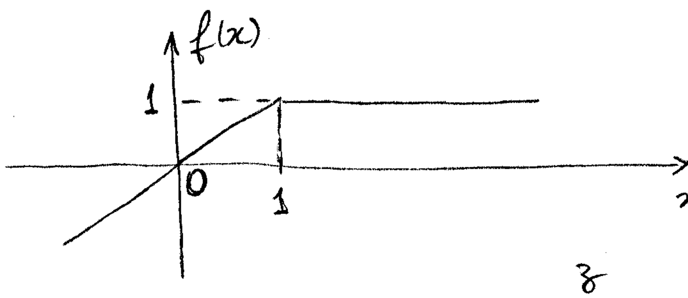
Good Luck !

Question	1	2	3	4	5	6	7	8
Score								

Question	9	10	11	12	Total
Score					

1) (10 points) Find the function given by $H(z) = \int_0^z f(x)\,dx$, $z \geq 0$, where

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



For $0 \leq z < 1$, $H(z) = \int_0^z x\,dx = \left.\frac{x^2}{2}\right]_0^z = \frac{z^2}{2} - 0 = \frac{z^2}{2}$

For $z \geq 1$, $H(z) = \int_0^1 x\,dx + \int_1^z 1\,dx = \left[\frac{x^2}{2}\right]_0^1 + \left[x\right]_1^z = \frac{1}{2} + z - 1$

$= z - \frac{1}{2}$

Answer: $\boxed{H(z) = \begin{cases} z^2/2 & \text{for } 0 \leq z < 1 \\ z - \frac{1}{2} & \text{for } z \geq 1 \end{cases}}$

2) (10 points) Compute the improper integral

$$\int_0^\infty \frac{x}{(1+x^2)^3}\,dx$$

$$\int_0^\infty \frac{x}{(1+x^2)^3}\,dx = \lim_{b\to\infty} \int_0^b \frac{x}{(1+x^2)^3}\,dx = \lim_{b\to\infty} \int_1^{1+b^2} \frac{1}{2}\frac{1}{u^3}\,du$$

$u = 1 + x^2$

$du = 2x\,dx$

$u(0) = 1$

$u(b) = 1 + b^2$

$$= \lim_{b\to\infty} \left[\frac{1}{2}\left(-\frac{1}{2}u^{-2}\right)\right]_1^{1+b^2} = \lim_{b\to\infty} \left(\frac{-1}{4(1+b^2)} + \frac{1}{4}\right) = \frac{1}{4}$$

3) (10 points) Find the volume of the solid obtained by rotating the region bounded by the curve given below about the x-axis.

$$y = \frac{1}{\sqrt{(x+1)(x+2)}}, \quad 0 \le x \le 1$$

$$V = \int_0^1 \pi y^2 dx = \pi \int_0^1 \frac{1}{(x+1)(x+2)} dx$$

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{(A+B)x + 2A+B}{(x+1)(x+2)} \Rightarrow \begin{cases} A+B=0 \\ 2A+B=1 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$V = \pi\left(\int_0^1 \frac{1}{x+1} dx - \int_0^1 \frac{1}{x+2} dx\right) = \pi\left\{\left[\ln|x+1|\right]_0^1 - \left[\ln|x+2|\right]_0^1\right\}$$

$$= \pi\left\{\ln 2 - \cancel{\ln 1} - (\ln 3 - \ln 2)\right\} = \pi(2\ln 2 - \ln 3) = \boxed{\pi \ln\left(\frac{4}{3}\right)}$$

4) (10 points) Solve the differential equation with initial condition below.

$$\frac{dy}{dx} = \sin(x)\, e^{-y} \text{ with } y = 0 \text{ when } x = 0.$$

$$\int e^y dy = \int \sin(x) dx \Rightarrow e^y = -\cos(x) + C$$

$$C = ? \quad e^0 = -\cos(0) + C \Rightarrow 1 = -1 + C \Rightarrow C = 2$$

$$e^y = -\cos(x) + 2 \Rightarrow \boxed{y = \ln(2 - \cos(x))}$$

5) (10 points) Suppose that the fish population in a lake evolves according to the equation

$$\frac{dN}{dt} = 3N\left(1 - \frac{N}{1500}\right) - hN,$$

where h is the fishing rate. What condition on h is needed for the existence of an equilibrium with positive population size?

$$\frac{dN}{dt} = g(N) = 3N\left(1 - \frac{N}{1500}\right) - hN = -\frac{N^2}{500} + (3-h)N$$

$$g(N) = 0 \Rightarrow -\frac{N^2}{500} + (3-h)N = 0 \Rightarrow \begin{cases} N = 0 \\ \text{or} \\ -\frac{N}{500} + 3 - h = 0 \therefore N = 500(3-h) \end{cases}$$

To have a positive equilibrium, must have

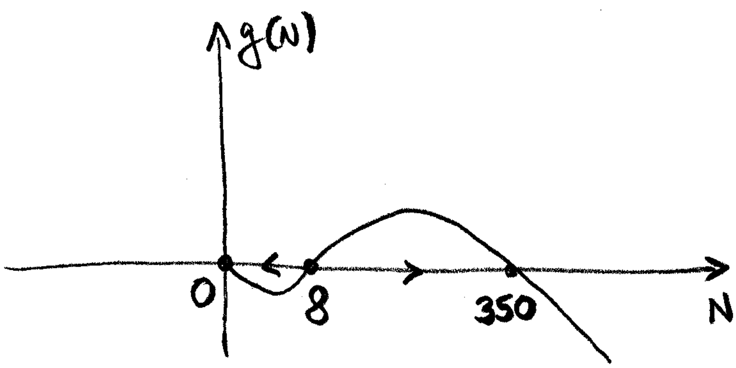
$$500(3-h) > 0 \therefore 3 - h > 0 \therefore \boxed{h < 3}$$

6) (10 points) Denote by $N(t)$ the size of a population at time t, and assume that

$$\frac{dN(t)}{dt} = 0.4\,N\,(N-8)\left(1 - \frac{N}{350}\right) \quad \text{for } t \geq 0.$$

Find all the equilibria and indicate which ones are stable and which ones are unstable.

Equilibria : $0.4\,N(N-8)\left(1 - \frac{N}{350}\right) = 0$



$$\Rightarrow \begin{cases} N = 0 & \text{stable} \\ N = 8 & \text{unstable} \\ N = 350 & \text{stable} \end{cases}$$

7) (10 points) Find c, so that the function $f(x) = c\,x\,e^{-x^2}$, $x \geq 0$ becomes a density function

$$1 = \int_0^\infty c x e^{-x^2} dx = c \lim_{b\to\infty} \int_0^b x e^{-x^2} dx = c \lim_{b\to\infty} \int_0^{b^2} \frac{1}{2} e^{-u} du$$

$u = x^2$

$du = 2x dx$

$u(0) = 0$

$u(b) = b^2$

$$= c \lim_{b\to\infty} \left[\frac{e^{-u}}{-2}\right]_0^{b^2} = c \lim_{b\to\infty}\left\{\frac{e^{-b^2}}{-2} + \frac{1}{2}\right\} = \frac{c}{2}$$

$$\frac{c}{2} = 1 \Rightarrow \boxed{c = 2}$$

8) (10 points) Suppose that a quantity X is distributed according to the density function

$$f(x) = \begin{cases} 6\,x^5 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean of X.

$$\text{mean} = \int_0^1 x f(x) dx = \int_0^1 x \cdot 6x^5 dx = \int_0^1 6x^6 dx$$

$$= 6 \cdot \frac{x^7}{7}\Big]_0^1 = \boxed{\frac{6}{7}}$$

9) (10 points) Where is the function $f(x,y) = \sqrt{x^2 + e^{xy}}$ continuous?

$f(x,y) = g(h(x,y))$ where $g(z) = \sqrt{z}$, $h(x,y) = x^2 + e^{xy}$

So $f(x,y)$ is a composition of continuous functions and therefore continuous on its domain.

D = Domain of $f(x,y)$: $x^2 + e^{xy} \geq 0$. We know that for every x, y $x^2 \geq 0$, $e^{xy} > 0$.

Therefore : $D = \mathbb{R}^2$.

Answer : everywhere on $\mathbb{R}^2$

10) (10 points) Compute $\frac{\partial^2 h(x,y)}{\partial x \, \partial y}$, where $h(x,y) = \ln(xy + \sin(y))$.

$$\frac{\partial h(x,y)}{\partial y} = \frac{1}{xy + \sin(y)} \cdot \frac{\partial}{\partial y}(xy + \sin(y)) = \frac{x + \cos(y)}{xy + \sin(y)}$$

$$\frac{\partial^2 h(x,y)}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{x + \cos(y)}{xy + \sin(y)}\right) = \frac{1 \cdot (xy + \sin(y)) - (x + \cos(y)) \cdot y}{(xy + \sin(y))^2}$$

$$= \frac{\sin(y) - y\cos(y)}{(xy + \sin(y))^2}$$

11) (10 points) Find the equation of the tangent plane to the graph of $g(x, y) = 2x^3 + y^2$ at $(x, y) = (1, 2)$.

$\frac{\partial g}{\partial x} = 6x^2$ $\quad \frac{\partial g}{\partial x}(1,2) = 6$ $\quad \frac{\partial g}{\partial y} = 2y$ $\quad \frac{\partial g}{\partial y}(1,2) = 4$

Tangent plane : $z - g(1,2) = 6(x-1) + 4(y-2)$

$z - 6 = 6x - 6 + 4y - 8$

$6x + 4y - z - 8 = 0$

12) (10 points) Compute

$$\int \sin^2(x)\, dx$$

$\int \sin^2(x)\, dx = \int \underbrace{\sin(x)}_{f(x)} \cdot \underbrace{\sin(x)}_{g'(x)}\, dx$ integration by parts

$f'(x) = \cos(x)$
$g(x) = -\cos(x)$

$= \sin(x)(-\cos(x)) - \int \cos(x)(-\cos(x))\, dx$

$= -\sin(x)\cos(x) + \int \cos^2(x)\, dx$

$= -\sin(x)\cos x + \int (1 - \sin^2(x))\, dx = -\sin(x)\cos(x) + x - \int \sin^2(x)\, dx$

$\therefore\ 2\int \sin^2(x)\, dx = -\sin(x)\cos(x) + x \ \therefore\ \boxed{\int \sin^2(x)\, dx = \frac{-\sin(x)\cos(x) + x}{2} + C}$