MATH 3A (Fall 2001)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names: Solutions

Signature:

UCLA id number (if you are an extension student, say so):

Circle the discussion section in which you are enrolled:

2A (Tue. 9am, Rahul) 2B (Thur. 9am, Rahul) 2C (Tue. 9am, Tim)

2D (Thur. 9am, Tim) 2E (Tue. 9am, Brian) 2F (Thur. 9am, Brian)

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam.

Good Luck !

Question	1	2	3	4	5	6	7	8
Score								

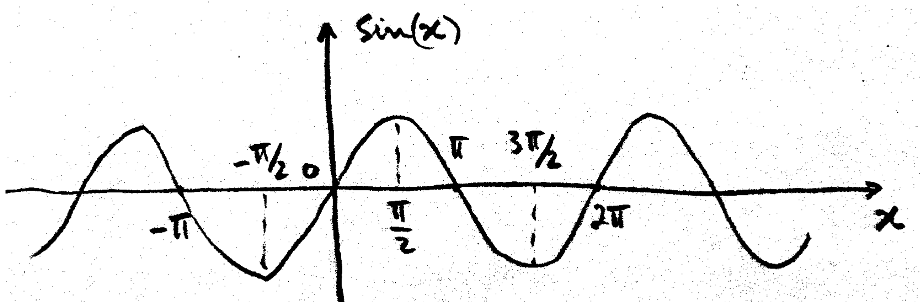
Question	9	10	11	12	13	14	15	16	Total
Score									

1) (10 points) Compute $\lim_{x\to\infty} \frac{1+e^{2x}}{9e^{2x}+4e^{x}}$

$$\lim_{x\to\infty} \frac{1+e^{2x}}{9e^{2x}+4e^{x}} = \lim_{x\to\infty} \frac{e^{2x}}{9e^{2x}} \frac{e^{-2x}+1}{1+4e^{-x}}$$

$$= \lim_{x\to\infty} \frac{1}{9} \lim_{x\to\infty} \frac{e^{-2x}+1}{1+4e^{-x}} = \frac{1}{9} \times 1 = \boxed{\frac{1}{9}}$$

2) (10 points) Sketch the graph of $\sin(x)$ and based on the sketch find the value of $\lim_{x\to\infty} \sin(x)$.



$\lim_{x\to\infty} \sin(x)$ does not exist, since $\sin(x)$ does not approach any value as $x \to \infty$.

3) (10 points) Compute b so that $f(x)$ below becomes continuous at $x = 1$ for a proper choice of c.

$$f(x) = \begin{cases} \frac{x^2-3x+2}{b(x-1)} & \text{if } x < 1 \\ \frac{\sin(x-1)}{x-1} & \text{if } x > 1 \\ c & \text{if } x = 1 \end{cases}$$

$$\lim_{x \to 1^-} \frac{x^2-3x+2}{b(x-1)} \overset{\text{L'Hopital } \frac{0}{0}}{=} \lim_{x \to 1^-} \frac{2x-3}{b} = -\frac{1}{b}$$

$$\lim_{x \to 1^+} \frac{\sin(x-1)}{x-1} = 1 \quad \left(\text{Since } \lim_{x \to 1^+} (x-1) = 0, \ \lim_{z \to 0} \frac{\sin z}{z} = 1 \right)$$

Want $-\frac{1}{b} = 1 \quad \therefore \quad \boxed{b = -1}$

4) (10 points) You want to find a solution to the equation $x^3 - 2 = 0$ in the interval $[0, 2]$, using the bisection method. After one step of this method, in which interval will you be looking for a a solution to the equation?

$f(x) = x^3 - 2$ start from $[0,2]$ midpoint: 1

$f(0) = -2$ negative

$f(2) = 8-2=6$ positive

$f(1) = 1-2=-1$ negative

So 1 replaces 0, and the interval is now:

Answer: $[1, 2]$

5) (10 points) Define the derivative of a funtion $f(x)$ at $x = c$.

$$\frac{df(c)}{dx} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

provided that the limit exists.

6) (10 points) Find the equation of the tangent line to the graph of the function $f(x) = \frac{x^2-5}{\sqrt{2x^3+1}}$, through the point $(0, -5)$.

$$f'(x) = \frac{2x\sqrt{2x^3+1} - (x^2-5)\frac{1}{2\sqrt{2x^3+1}} \cdot 6x^2}{2x^3+1}$$

$$f'(0) = 0$$

tangent line through $(0,-5)$:

$$y - (-5) = f'(0)(x - 0)$$

$\therefore$ $\boxed{y = -5}$

7) (10 points) Compute the first and second derivatives of the function $f(x) = \log_5(x^2+1)$.

$$f(x) = \frac{\ln(x^2+1)}{\ln 5}$$

chain rule

$$f'(x) = \frac{1}{\ln 5} \cdot \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) = \frac{2x}{\ln 5\,(x^2+1)}$$

quotient rule

$$f''(x) = \frac{2(x^2+1) - 2x(2x)}{\ln 5\,(x^2+1)^2} = \frac{2(1-x^2)}{\ln 5\,(x^2+1)^2}$$

8) (10 points) Find $\frac{dy}{dx}$ by implicit differentiation, when $\sqrt{xy} = x^2+1$. (Recall that y is a function of x and that the answer will be given in terms of x and y.)

$$\frac{d}{dx}\sqrt{xy} = \frac{d}{dx}(x^2+1)$$

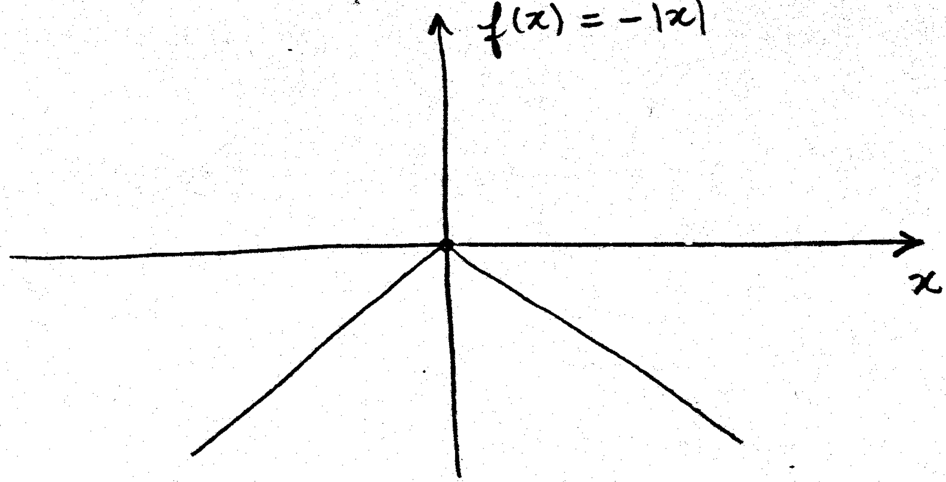
$$\frac{1}{2\sqrt{xy}}\frac{d}{dx}(xy) = 2x$$

$$\frac{y + xy'}{2\sqrt{xy}} = 2x$$

$$xy' = 4x\sqrt{xy} - y$$

$$y' = 4\sqrt{xy} - \frac{y}{x}$$

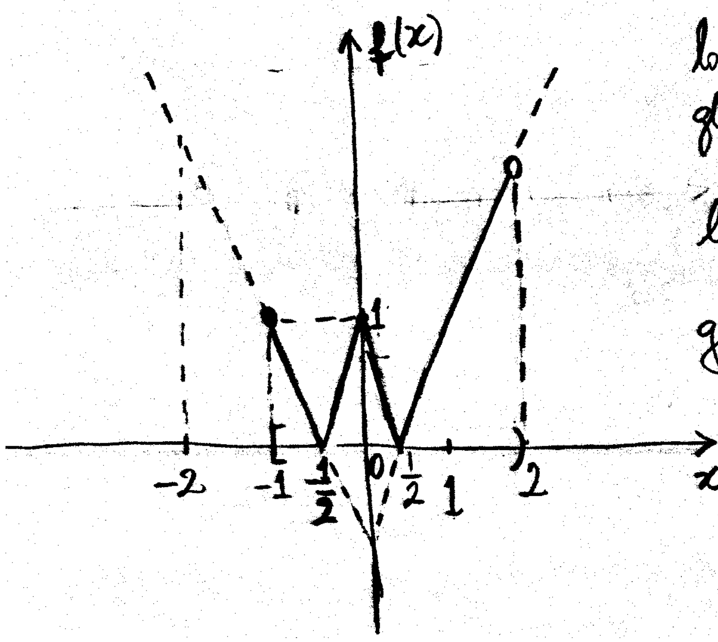
9) (10 points) Sketch the graph of a function $f(x)$, defined for all values of $x \in \mathbb{R}$, that clearly shows that a local maximum can occur at a value $x = c$ which does not satisfy $f'(c) = 0$.



0 is a local maximum, but $f'(0)$ does not exist.

10) (10 points) Sketch the graph of the function below and list all its local maxima, local minima, global maxima, and global minima.

$$f(x) = |\,|2x| - 1\,|, \quad -1 \leq x < 2.$$



local max : $-1, 0$

global max : None

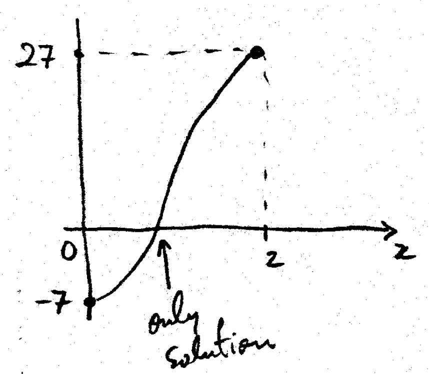
local min : $-\frac{1}{2}, \frac{1}{2}$

global min : $-\frac{1}{2}, \frac{1}{2}$

11) (10 points) How many solutions does the equation $x^5+x-7=0$ have in the interval $[0,2]$? (Hint: Sketch the graph of the function $f(x)=x^5+x-7$, $0 \le x \le 2$.)

$f'(x) = 5x^4+1 > 0$, so $f(x)$ is increasing

$f(0) = -7$, $f(2) = 2^5+2-7 = 27$



Answer: One solution

12) (10 points) Find the inflection points of $f(x)=\arctan(x-5)$. (Hint: Recall that $\frac{d}{dx}\arctan(x)=\frac{1}{1+x^2}$.)

$$f'(x) = \frac{1}{1+(x-5)^2} \cdot \frac{d}{dx}(x-5) = \frac{1}{1+(x-5)^2}$$

$$f''(x) = \frac{-2(x-5)}{(1+(x-5)^2)^2} = \frac{10-2x}{(1+(x-5)^2)^2}$$

$$f''(x) = 0 \Rightarrow 10-2x=0 \Rightarrow x=5$$

x		5	
sign $f''(x)$	+ + + + + + +		− − − − − − − − −

Answer: So 5 is the only inflection point

13) (10 points) Sketch the graph of $f(x) = xe^{-x}$, indicating the local maxima, local minima and inflection points. Be sure that your sketch indicates the correct limits as $x \to -\infty$ and $x \to +\infty$.

$$f'(x) = e^{-x} + x(-e^{-x}) = e^{-x}(1-x)$$

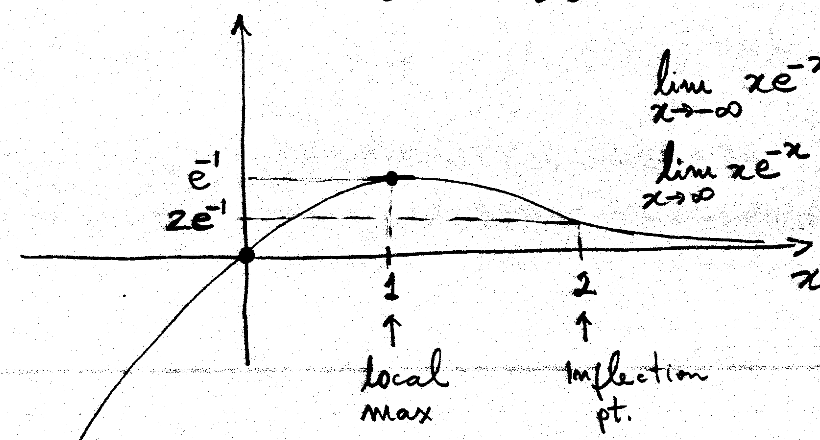
$$f'(x) = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$$

$$f''(x) = (-e^{-x})\cdot(1-x) + e^{-x}\cdot(-1)$$

$$= e^{-x}(-1 + x - 1) = e^{-x}(x-2)$$

$$f''(x) = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$$

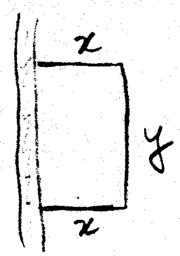
x		1		2	
sign $f'(x)$	+ + + + + + + +		− − −		− − − −
sign $f''(x)$	− − − − −		− − −		+ + + +
$f(x)$		e^{-1}		$2e^{-2}$	



$$\lim_{x \to -\infty} xe^{-x} = -\infty$$

$$\lim_{x \to \infty} xe^{-x} = \lim_{x \to \infty} \frac{x}{e^x} \overset{\frac{\infty}{\infty}\text{ l'Hopital}}{=} \lim_{x \to \infty} \frac{1}{e^x} = 0$$

14) (10 points) A rectangular field is bounded on one side by a river and on the other three sides by a fence. Find the maximal area that can be enclosed in this way if the fence has total length 400 ft.

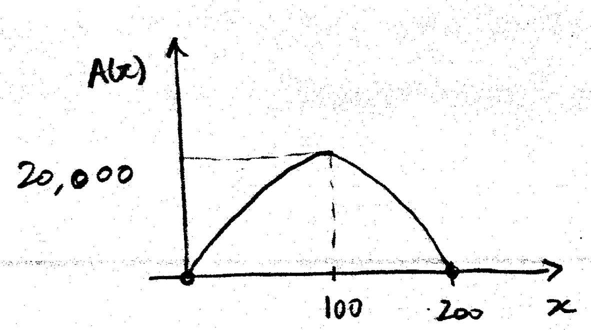


$$\begin{cases} 2x + y = 400 \\ x \cdot y = A \end{cases} \qquad 0 < x < 200,\ 0 < y < 400 \qquad A \text{ maximal} = ?$$

$$y = 400 - 2x$$

$$A(x) = x \cdot (400 - 2x) = 400x - 2x^2$$
$$0 < x < 200$$

$$A'(x) = 400 - 4x$$
$$A'(x) = 0 \implies 400 - 4x = 0 \implies x = 100 \text{ ft}$$
$$A''(x) = -4 < 0 \implies x = 100 \text{ ft is local max}$$
$$\text{maximal } A = 400 \times 100 - 2 \times (100)^2$$
$$= 40{,}000 - 20{,}000$$
$$= \boxed{20{,}000 \text{ sq ft}}$$



15) (10 points) Compute $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\exp\left(x\ln\left(1+\frac{1}{x}\right)\right)$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\overset{\text{L'Hopital } \frac{0}{0}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

So $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \exp(1) = \boxed{e}$

16) (10 points) Suppose that when a certain substance is injected in the vein, its blood amount (in miligrams) a time t (in seconds) after the injection is given by $f(t)$, which satisfies

$$\frac{df(t)}{dt} = \exp(-0.1\,t).$$

Find $f(t), t \geq 0$, if we know that $\lim_{t\to\infty} f(t) = 12$.

$$f(t) = -\frac{1}{0.1}\exp(-0.1\,t) + C$$

$$\lim_{t\to\infty} f(t) = 12 \Rightarrow C = 12$$

So $f(t) = 12 - \dfrac{e^{-0.1t}}{0.1} = \boxed{12 - 10e^{-0.1t}}$