MATH 3A (Winter 2002)
Instructor: Roberto Schonmann
Final Exam

Last Name: Solutions

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Circle the discussion section in which you are enrolled:

1A (Tue. 8am, Stephanie) 1B (Thur. 8am, Stephanie)

1C (Tue. 8am, Kathleen) 1D (Thur. 8am, Kathleen)

1E (Thur. 8am, Sumni)

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit close to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	8
Score								

Question	9	10	11	12	13	Total
Score						

1) (10 points) Compute $\lim_{x\to 0}\frac{e^{2x}-1}{e^{x}-1}$

<u>1st Solution</u>: $\lim_{x\to 0}\frac{e^{2x}-1}{e^{x}-1} = \lim_{x\to 0}\frac{(e^{x})^2-1^2}{e^{x}-1}$

$= \lim_{x\to 0}\frac{\cancel{(e^{x}-1)}(e^{x}+1)}{\cancel{e^{x}-1}} = \lim_{x\to 0}(e^{x}+1) = \boxed{2}$

<u>2nd Solution</u>: $\lim_{x\to 0}\frac{e^{2x}-1}{e^{x}-1} \overset{\text{L'Hopital}(\frac{0}{0})}{=} \lim_{x\to 0}\frac{2e^{2x}}{e^{x}} = \frac{2\times 1}{1} = \boxed{2}$

2) (10 points) Explain how the sandwich theorem can be used to compute $\lim_{x\to\infty} e^{-x}\cos(x)$, and find this limit.

We have $\quad -1 \leq \cos(x) \leq 1 \quad$ and $e^{-x}>0$

So $\quad (-1)\cdot e^{-x} \leq e^{-x}\cos(x) \leq 1\cdot e^{-x}$

$$-e^{-x} \leq e^{-x}\cos(x) \leq e^{-x}$$

We know that $\lim_{x\to\infty} e^{-x} = 0$. So also $\lim_{x\to\infty} -e^{-x} = 0$

And by the sandwich thm $\lim_{x\to\infty} e^{-x}\cos x = \boxed{0}$

3) (10 points) Compute b so that $f(x)$ below becomes continuous at $x = 1$.

$$f(x) = \begin{cases} \frac{x^2-2x+1}{x-1} & \text{if } x < 1 \\ b + \cos(x-1) & \text{if } x \geq 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x^2 - 2x + 1}{x-1} = \lim_{x \to 1^-} \frac{(x-1)^2}{x-1}$$

$$= \lim_{x \to 1^-} (x-1) = 0$$

$$f(1) = b + \cos(1-1) = b + \cos(0) = b+1$$

So we must have $0 = b+1 \quad \therefore \quad \boxed{b = -1}$

4) (10 points) Compute the limit

$$\lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$\lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \left\{ \sin(x) \frac{\cos(h) - 1}{h} + \cos(x) \frac{\sin(h)}{h} \right\}$$

$$= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \boxed{\cos(x)}$$

5) (10 points) Find the equation of the tangent line to the graph of the function $f(x) = (\ln(x))^2$, through the point $(e, 1)$.

$$f'(x) = 2\ln(x)\frac{d}{dx}\ln(x) = 2\ln(x)\cdot\frac{1}{x} = \frac{2\ln(x)}{x}$$

slope of tangent line $= m = f'(e) = \frac{2\ln(e)}{e} = \frac{2}{e}$

tangent line: $y - 1 = m(x - e)$

$$\therefore\ y - 1 = \frac{2}{e}(x - e) = \frac{2}{e}x - 2\ \therefore\ \boxed{y = \frac{2}{e}x - 1}$$

6) (10 points) Compute the derivative of the function $f(x) = \exp(\sqrt{x}\cos(x))$.

chain rule

$$f'(x) = \exp(\sqrt{x}\cos(x)) \cdot \frac{d}{dx}(\sqrt{x}\cos(x))$$

$$= \exp(\sqrt{x}\cos(x)) \cdot \frac{d}{dx}(x^{1/2}\cos(x))$$

product rule

$$= \exp(\sqrt{x}\cos(x)) \cdot \left(\frac{1}{2}x^{-1/2}\cos(x) + x^{1/2}(-\sin(x))\right)$$

$$= \exp(\sqrt{x}\cos(x))\left(\frac{\cos(x)}{2\sqrt{x}} - \sqrt{x}\sin(x)\right)$$

7) (10 points) Let $f(x) = x^7 + x + 2$. Find $\frac{d}{dx} f^{-1}(4)$.

$f'(x) = 7x^6 + 1$ $\qquad$ $f^{-1}(4) = 1$ (by trial-and-error)

$$\frac{d}{dx} f^{-1}(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{7(f^{-1}(4))^6 + 1}$$

$$= \frac{1}{7 \times 1^6 + 1} = \boxed{\frac{1}{8}}$$

8) (10 points) Use a linear approximation to find an approximate value for $\tan(0.02)$.

$\tan(0) = 0$

$$\frac{d}{dx} \tan(x) = \frac{1}{\cos^2(x)} \qquad \frac{d}{dx} \tan(0) = \frac{1}{\cos^2(0)} = 1$$

linear approximation at 0:

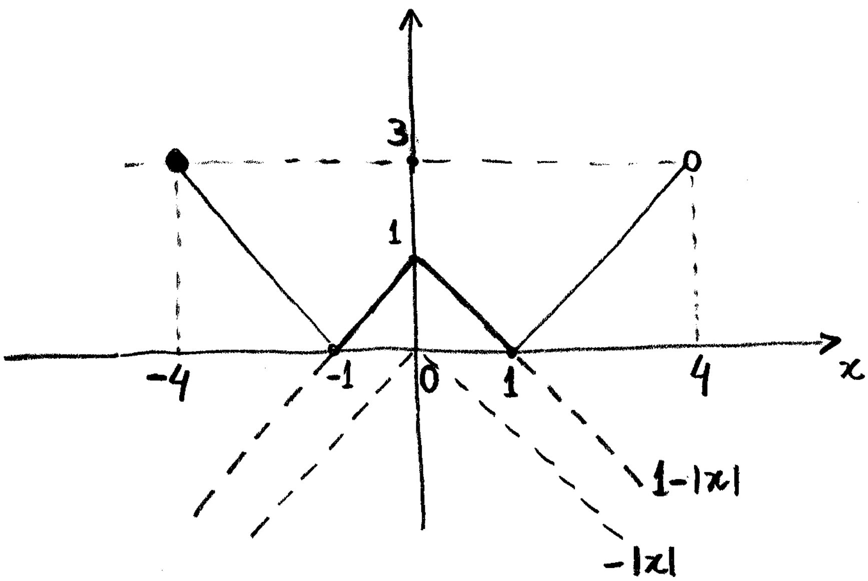
$$L(x) = \tan(0) + \tan'(0) \cdot (x - 0)$$
$$= 0 + 1 \cdot (x - 0) = x$$

$$f(0.02) \cong L(0.02) = \boxed{0.02}$$

9) (10 points) Graph

$$f(x) = |1 - |x||, \quad -4 \le x < 4$$

and determine all the local maxima, local minima, global maxima and global minima.



loc. max : $-4, 0$
loc. min : $-1, 1$
gl. max : -4
gl. min : $-1, 1$

10) (10 points) Find the inflection points of $f(x) = x^{100}$.

$f'(x) = 100x^{99}$ $\qquad$ $f''(x) = 9900x^{98}$

$f''(x) = 0 \Rightarrow x = 0$

x		0	
$f''(x)$	$+$	0	$+$

no inflection point

11) (10 points) Find the local maxima, local minima, global maxima and global minima of $f(x) = \frac{1}{4}x^3 + x^2 + x$.

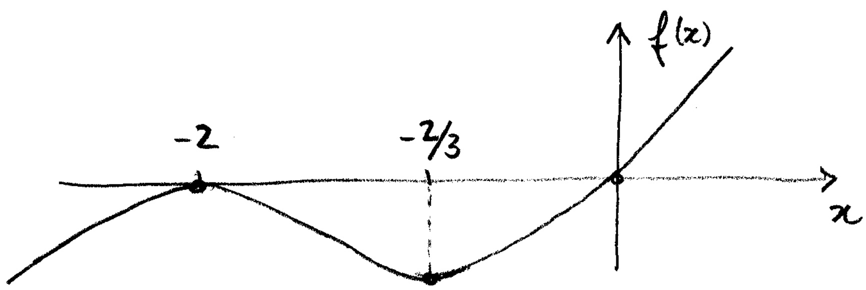
$$f'(x) = \frac{3}{4}x^2 + 2x + 1 \qquad f'(x) = 0 \Rightarrow x = \frac{-2 \pm \sqrt{4-3}}{6/4} = \frac{2}{3}(-2 \pm 1) = \begin{cases} -\frac{2}{3} \\ -2 \end{cases}$$

$$f'(x) = \frac{3}{4}(x + 2/3)(x+2)$$

$$\lim_{x \to \infty} f(x) = \infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

x		-2		$-2/3$	
$f'(x)$	$+$	0	$-$	0	$+$



$\boxed{\begin{array}{l} \text{loc. max} : -2 \\ \text{loc. min} : -2/3 \\ \text{glob. max} : \text{None} \\ \text{glob. min} : \text{None} \end{array}}$

12) (10 points) Suppose that the growth rate of a population at time t undergoes seasonal fluctuations in size according to

$$\frac{dN}{dt} = 3\sin(2\pi t),$$

where t is measured in years and $N(t)$ denotes the size of the population at time t. If $N(0) = 5$ (measured in thousands), find $N(t)$.

$$N(t) = \frac{-3}{2\pi}\cos(2\pi t) + C$$

$$N(0) = 5 \Rightarrow 5 = -\frac{3}{2\pi} \cdot 1 + C \therefore C = 5 + \frac{3}{2\pi}$$

$$\boxed{N(t) = -\frac{3}{2\pi}\cos(2\pi t) + 5 + \frac{3}{2\pi}}$$

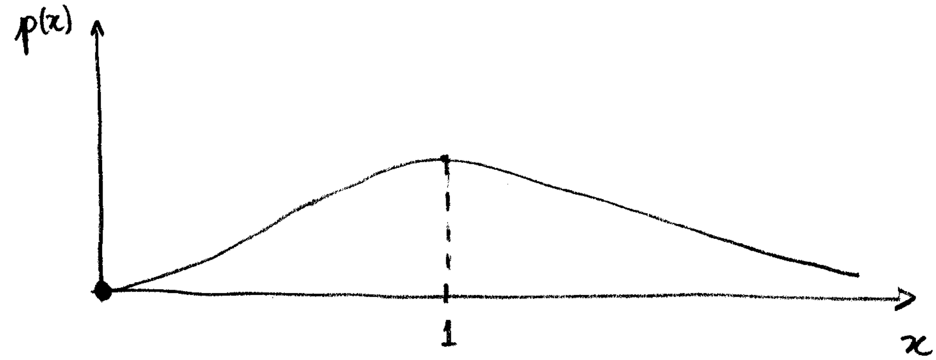
13) (10 points) The probability that a certain drug will have a beneficial effect on a disease, without causing any side-effect, is given by $p(x) = xe^{-x}$, where $x \geq 0$ is the amount used of the drug. What amount x maximizes this probability?

$$\frac{dp(x)}{dx} = e^{-x} + x(-e^{-x}) = e^{-x}(1-x)$$

Domain : $[0,\infty)$

$p'(x) = 0 \Rightarrow 1-x=0 \Rightarrow x=1$

x	0		1	
$p'(x)$		$+$	0	$-$



$\boxed{\text{Answer : } x=1}$