MATH 33B (Lecture 1, Winter 2004)
Instructor: Roberto Schonmann
Midterm Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Circle the discussion section in which you are enrolled:

1A (Tue. 9am, Yiannis)   1B (Thur. 9am, Yiannis)

1C (Tue. 9am, Cathy)   1D (Thur. 9am, Cathy)

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. In questions where there is a "yes or no" answer, the grading is always based on the explanation rather than on the answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	Total
Score								

1) (10 points) Express the number 0.2525252525... as a ratio between two integers.

$$\frac{2}{10} + \frac{5}{100} + \frac{2}{10 \times 100} + \frac{5}{100 \times 100} + \frac{2}{10 \times (100)^2} + \frac{5}{100 \times (100)^2} + \cdots$$

$$= \frac{2}{10} \left\{ 1 + \frac{1}{100} + \frac{1}{(100)^2} + \cdots \right\}$$

$$+ \frac{5}{100} \left\{ 1 + \frac{1}{100} + \frac{1}{(100)^2} + \cdots \right\}$$

$$= \frac{2/10}{1 - 1/100} + \frac{5/100}{1 - 1/100} = \frac{20}{100-1} + \frac{5}{100-1}$$

$$= \boxed{\frac{25}{99}}$$

2) (10 points) Does the series below converge? Explain your answer carefully (no credit will be given if the explanation is not correct).

$$\sum_{n=1}^{\infty} \left[\frac{1}{(n+1)n}\right]^{1/3}$$

$$u_n = \left[\frac{1}{(n+1)n}\right]^{1/3} \geq \left[\frac{1}{(n+1)^2}\right]^{1/3} = \frac{1}{(n+1)^{2/3}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^{2/3}} = \sum_{k=2}^{\infty} \frac{1}{k^{2/3}} \text{ diverges}$$

(p-series, $p = 2/3 < 1$)

By comparison: $\displaystyle\sum_{n=1}^{\infty} \left[\frac{1}{(n+1)n}\right]^{1/3}$ diverges

3) (10 points) Does the series below converge? Explain your answer carefully (no credit will be given if the explanation is not correct).

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

Ratio test : $\left| \frac{u_{n+1}}{u_n} \right| = \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}$

$$= \frac{\cancel{(2n)!}(2n+1)(2n+2)}{\cancel{(n!)^2} \cdot (n+1)^2} \quad \frac{\cancel{(n!)^2}}{\cancel{(2n)!}} = \frac{4n^2+6n+2}{n^2+2n+1}$$

$$\xrightarrow[\text{as } n \to \infty]{} 4 > 1$$

So series diverges

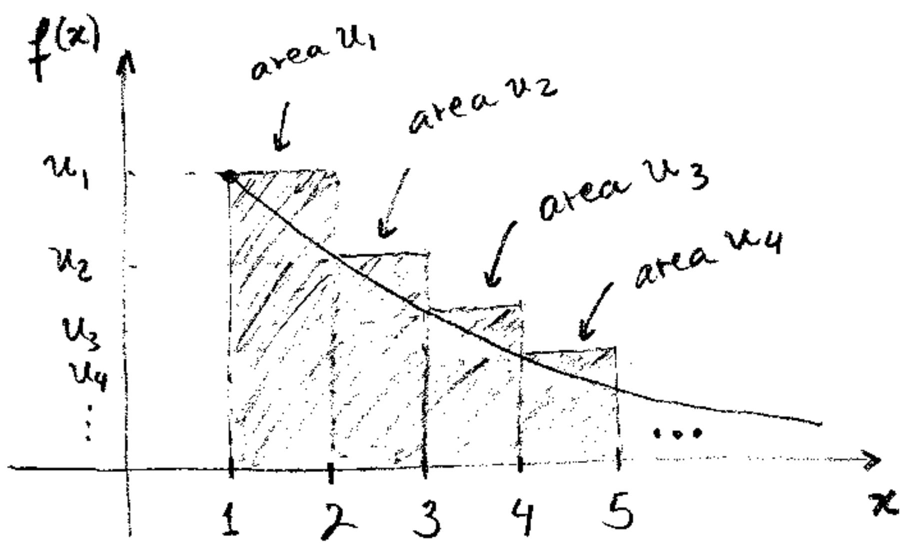
4) (10 points) One of the statements of the Integral Test is the following. Suppose that $f(x)$, $x \geq 1$, is a continuous, non-negative, and non-increasing function; suppose also that $u_n = f(n)$, $n = 1, 2, 3, ...$ Then if

$$\int_1^\infty f(x)\,dx = \infty,$$

we can conclude that the series

$$\sum_{n=1}^\infty u_n$$

diverges. Draw the picture that explains this statement and indicate clearly how the series above is related to the picture and why the result above is explained by the picture.



$$\sum_{n=1}^\infty u_n = (\text{shaded area}) \geq (\text{area below graph of } f(x)) = \int_1^\infty f(x)\,dx = \infty$$

So $\sum_{n=1}^\infty u_n$ diverges by comparison.

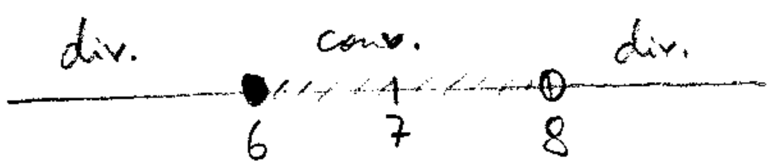
5) (10 points) For which values of x does the power series below converge? Do not forget to consider the end-points. Explain your answer carefully (no credit will be given if the explanation is not correct).

$$\sum_{n=1}^{\infty} \frac{(x-7)^n}{n}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x-7)^{n+1}}{n+1} \cdot \frac{n}{(x-7)^n}\right| = \frac{n}{n+1}\,|x-7|$$

$$\xrightarrow[\text{as } n\to\infty]{} |x-7| = \rho$$

$\rho = 1$: $|x-7| = 1 \Rightarrow x-7 = \pm 1 \Rightarrow x = 8$ or 6



$\rho < 1 \Rightarrow 6 < x < 8$ converges

$\rho > 1 \Rightarrow x < 6$ or $x > 8$ diverges

end-points : $x = 6 \to \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (alternating series)

$x = 8 \to \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

Answer : Converges when $6 \le x < 8$.

6) (10 points) Suppose that $f, f', ..., f^{(n)}, f^{(n+1)}$ are continuous on an interval containing a and b. What does the Taylor Theorem with Derivative form of the Remainder say about the remainder defined below?

$$R_n(a,b) \;=\; f(b) \;-\; \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(b-a)^k\right)$$

$$R_n(a,b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}$$

for some ξ between a and b.

7) (10 points) Find the sum of the series

$$\sum_{k=1}^{\infty} k\,x^k, \quad \text{for } |x| < 1.$$

$$\sum_{k=1}^{\infty} k\,x^k = x\sum_{k=1}^{\infty} k\,x^{k-1} = x\sum_{k=1}^{\infty} \frac{d}{dx} x^k$$

$$= x\,\frac{d}{dx}\left(\sum_{k=1}^{\infty} x^k\right) = x\,\frac{d}{dx}\left(\frac{x}{1-x}\right)$$

$$= x\,\frac{(1-x) - x\cdot(-1)}{(1-x)^2} = \boxed{\frac{x}{(1-x)^2}}$$