**MATH 33B** (Lecture 1, Winter 2004)
Instructor: Roberto Schonmann
**Final Exam**

**Last Name:**

**First and Middle Names:**

**Signature:**

**UCLA id number (if you are an extension student, say so):**

**Circle the discussion section in which you are enrolled:**

1A (Tue. 9am, Yiannis) 1B (Thur. 9am, Yiannis)

1C (Tue. 9am, Cathy) 1D (Thur. 9am, Cathy)

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. In questions where there is a "yes or no" answer, the grading is always based on the explanation rather than on the answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

**Good Luck !**

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
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| Score | | | | | | | | |

| Question | 9 | 10 | 11 | 12 | 13 | 14 | Total |
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1) (10 points) Does the series below converge absolutely? Explain your answer carefully (no credit will be given if the explanation is not correct).

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)\sqrt{n}}$$

comparison

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(n+1)\sqrt{n}} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

$$< \infty \quad (\text{p-series}, p = 3/2 > 1)$$

Answer: Yes

2) (10 points) Find the Maclaurin series for $\cosh(x^4)$. Recall that the Maclaurin series for $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $\cosh(x) = (e^x + e^{-x})/2$.

$$e^{x^4} = \sum_{n=0}^{\infty} \frac{(x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{4n}}{n!}$$

$+$

$$e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!}$$

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$$e^{x^4} + e^{-x^4} = \sum_{k=0}^{\infty} 2 \frac{x^{8k}}{(2k)!}$$

$$\cosh(x) = \frac{e^{x^4} + e^{-x^4}}{2} = \sum_{k=0}^{\infty} \frac{x^{8k}}{(2k)!}$$

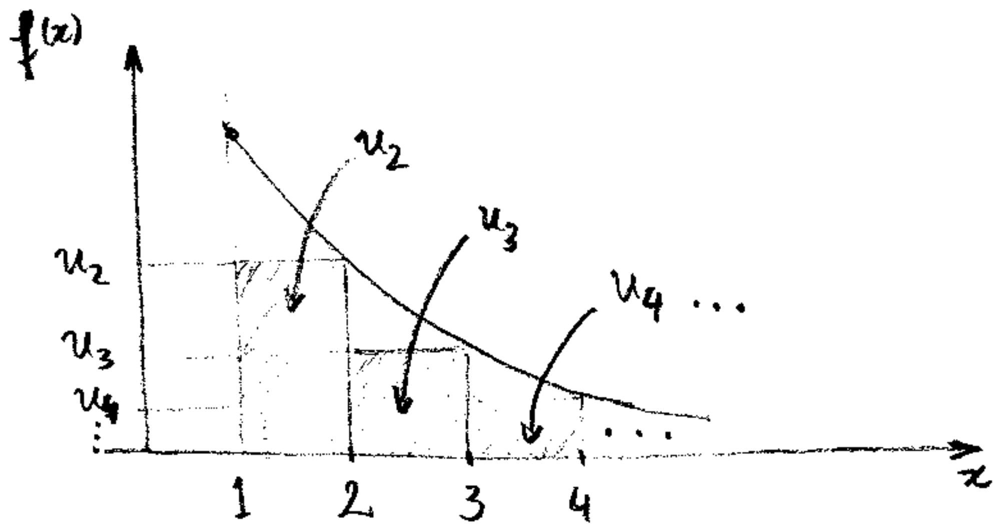
3) (10 points) One of the statements of the Integral Test is the following. Suppose that $f(x)$, $x \geq 1$, is a continuous, non-negative, and non-increasing function; suppose also that $u_n = f(n)$, $n = 1, 2, 3, \ldots$. Then if

$$\int_1^{\infty} f(x)\, dx$$

is finite, we can conclude that the series

$$\sum_{n=1}^{\infty} u_n$$

converges. Draw the picture that explains this statement and indicate clearly how the series above is related to the picture and why the result above is explained by the picture.




From picture $\quad u_2 + u_3 + u_4 + \ldots \leq \int_1^{\infty} f(x)\,dx$

So $\quad \sum_{n=1}^{\infty} u_n \leq u_1 + \int_1^{\infty} f(x)\,dx < \infty$

which means that $\sum_{n=1}^{\infty} u_n$ converges.

4) (10 points) For which values of $x$ does the power series below converge? Do not forget to consider the end-points. Explain your answer carefully (no credit will be given if the explanation is not correct).

$$\sum_{n=0}^{\infty} \frac{3^n (x-2)^n}{n!}$$

Ratio test $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{3^{n+1}(x-2)^{n+1}}{(n+1)!} \frac{n!}{3^n (x-2)^n}\right|$

$$= \frac{3}{n+1}|x-2| \xrightarrow[n\to\infty]{} 0 = \rho$$

Since $\rho < 1$, for every $x \in \mathbb{R}$ the series converges.

5) (10 points) Suppose that $f, f', ..., f^{(n)}, f^{(n+1)}$ are continuous on an interval containing $a$ and $b$. What does the Taylor Theorem with Derivative Form of the Remainder say about the remainder defined below?

$$R_n(a,b) \;=\; f(b) \;-\; \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(b-a)^k\right)$$

$$R_n(a,b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(b-a)^{n+1}$$

for some $\xi$ between a and b.

6) (10 points) Find the sum of the series

$$\sum_{k=1}^{\infty} k^2 x^k, \quad \text{for } |x| < 1.$$

(Hint: Find first the sum of the series $\sum_{k=1}^{\infty} k\, x^k$ and $\sum_{k=1}^{\infty} k(k-1)x^k$ )

$$\sum_{k=1}^{\infty} k x^k = x \sum_{k=1}^{\infty} k x^{k-1} = x \sum_{k=1}^{\infty} \frac{d}{dx}(x^k)$$

$$= x \frac{d}{dx}\left(\sum_{k=1}^{\infty} x^k\right) = x \frac{d}{dx}\left(\frac{x}{1-x}\right)$$

$$= x \frac{(1-x) - x(-1)}{(1-x)^2} = \frac{x}{(1-x)^2}$$

$$\sum_{k=1}^{\infty} k(k-1) x^k = x^2 \sum_{k=1}^{\infty} k(k-1) x^{k-2} = x^2 \sum_{k=1}^{\infty} \frac{d^2}{dx^2}(x^k)$$

$$= x^2 \frac{d^2}{dx^2}\left(\sum_{k=1}^{\infty} x^k\right) = x^2 \frac{d^2}{dx^2}\left(\frac{x}{1-x}\right) = x^2 \frac{d}{dx} \frac{1}{(1-x)^2}$$

$$= x^2 \frac{-2}{(1-x)^3} \cdot (-1) = \frac{2x^2}{(1-x)^3}$$

$$\sum_{k=1}^{\infty} k^2 x^k = \sum_{k=1}^{\infty} k(k-1)x^k + \sum_{k=1}^{\infty} k x^k = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$= \frac{2x^2 + x - x^2}{(1-x)^3} = \frac{x^2+x}{(1-x)^3}$$

7) (10 points) Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0, \\ x & \text{for } 0 \le x < \pi \ . \end{cases}$$

$n>0$: $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left\{\left[x\,\frac{\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}\,dx\right\} = \frac{1}{\pi}\left.\frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\,\frac{(-1)^n-1}{n^2} \qquad , \quad a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left\{\left[-\frac{x\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi(-1)^n}{n} + \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi}\right\} = \frac{(-1)^{n+1}}{n} + 0 = \frac{(-1)^{n+1}}{n}$$

Answer: $$\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{\pi}\,\frac{(-1)^n-1}{n^2}\cos(nx) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$$

8) (10 points) Find the solution $y(x)$ of the initial value problem:

$$x^2 y' + y^2 = 0, \qquad y(1) = 1.$$

Separation of variables:

$$\frac{y'}{y^2} = -\frac{1}{x^2}$$

$$\int \frac{dy}{y^2} = -\int \frac{dx}{x^2}$$

$$-\frac{1}{y} = +\frac{1}{x} + C$$

$$y = \frac{-1}{\frac{1}{x} + C}$$

$C = ?$ $\quad y(1) = 1 \Rightarrow 1 = \frac{-1}{1+C} \Rightarrow C = -2$

$$y(x) = \frac{1}{2 - \frac{1}{x}}$$

9) (10 points) Find the solution $y(x)$ of the initial value problem:

$$y' + \frac{2}{x}y = \frac{1}{x^2}, \qquad y(1) = 5.$$

First order linear

integrating factor : $e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = |x|^2 = x^2$

$y' x^2 + 2xy = 1$

$\frac{d}{dx}(y x^2) = 1$

$y x^2 = x + C$

$y = \frac{x + C}{x^2}$

$C = ?$ $\quad y(1) = 5 \Rightarrow 5 = \frac{1 + C}{1} \Rightarrow C = 4$

$y(x) = \frac{x + 4}{x^2}$

10) (10 points) Find the function $y(x)$ which satisfies the two differential equations below and also the condition $y(0) = 4$. (We did not solve exercises like this one in the course, but you should be able to do it with what you learned there.)

$$y'' + y' - 2y = 0, \qquad y'' - 8y' + 7y = 0.$$

$y'' + y' - 2y = 0 \longrightarrow \lambda^2 + \lambda - 2 = 0 \longrightarrow \lambda = \frac{-1 \pm \sqrt{9}}{2} = \begin{cases} 1 \\ -2 \end{cases}$

Solution : $y(x) = a e^{x} + b e^{-2x} \quad a, b \in \mathbb{R}$

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$y'' - 8y' + 7y = 0 \rightarrow \lambda^2 - 8\lambda + 7 = 0 \rightarrow \lambda = \frac{8 \pm \sqrt{64-28}}{2}$

$= \frac{8 \pm \sqrt{36}}{2} = \frac{8 \pm 6}{2} = \begin{cases} 7 \\ 1 \end{cases}$

Solution : $y(x) = c e^{7x} + d e^{x} \quad c, d \in \mathbb{R}$

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Solutions to both diff. eq : $y(x) = A e^{x}$ , $A \in \mathbb{R}$

$A = ?$ $\quad y(0) = 4 \Rightarrow A = 4$

$\boxed{\text{Answer: } y(x) = 4e^{x}}$

11) (10 points) Find the solution $y(x)$ of the initial value problem:

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 5.$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-16}}{2} = 2$$

General solution : $y(x) = ae^{2x} + bxe^{2x}$

$a, b = ?$ $\qquad y'(x) = 2ae^{2x} + be^{2x} + 2bxe^{2x}$

$$\begin{cases} 1 = y(0) = a \\ 5 = y'(0) = 2a + b \Rightarrow b = 5 - 2a = 3 \end{cases}$$

$$\boxed{\text{Answer : } y(x) = e^{2x} + 3xe^{2x}}$$

12) (10 points) Suppose that $y(x)$ satisfies the differential equation $y'' + y' + y = 0$. Compute $\lim_{x\to\infty} y(x)$.

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2} = \alpha + \beta i$$

$$y(x) = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$$

$$= e^{-x/2}\left(c_1 \cos\left(\sqrt{3}/2\, x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} x\right)\right)$$

$\lim_{x\to\infty} y(x) = 0$ since $e^{-x/2} \to 0$

and sin, cos oscilate between $-1$ and $+1$.

13) (10 points) Find a particular solution of

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x^4}, \qquad (x > 0),$$

given that two linearly independent solutions of the associated homogeneous equation are $y_1(x) = x$ and $y_2(x) = 1/x$.

$$y(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$$

$$\begin{cases} y_1(x)\, c_1'(x) + y_2(x)\, c_2'(x) = 0 \\ y_1'(x)\, c_1'(x) + y_2'(x)\, c_2'(x) = 1/x^4 \end{cases}$$

$$\begin{cases} x\, c_1'(x) + (1/x)\, c_2'(x) = 0 & \cdot 1 \\ c_1'(x) - (1/x^2)\, c_2'(x) = 1/x^4 & \cdot x \end{cases} \quad +$$

$$2x\, c_1'(x) = 1/x^3 \quad \therefore \quad c_1'(x) = \frac{1}{2x^4}$$

$$c_2'(x) = -x^2 c_1'(x) = -\frac{1}{2x^2}$$

$$\begin{cases} c_1(x) = \int c_1'(x)\, dx = \int \frac{1}{2} x^{-4} = -\frac{x^{-3}}{6} \\ c_2(x) = \int c_2'(x)\, dx = \int \left(-\frac{1}{2}\right) x^{-2} = \frac{x^{-1}}{2} \end{cases}$$

$$y(x) = -\frac{x^{-3}}{6}\, x + \frac{x^{-1}}{2} \cdot \frac{1}{x} = x^{-2}\left(-\frac{1}{6} + \frac{1}{2}\right) = \boxed{\frac{x^{-2}}{3}}$$

14) (10 points) Consider the initial value problem below for $x(t)$ and $y(t)$:

$$\begin{cases} x' = x + 2y + \cos(t) & (\mathrm{I}) \\ y' = y - 3x + e^{t-1} + t^2 & (\mathrm{II}) \end{cases}$$

$$x(0) = 1, \qquad y(0) = 4.$$

Find a second order differential equation for $x(t)$ and the corresponding initial values of $x(t)$ and $x'(t)$. (You are <u>not</u> asked to solve this initial value problem for $x(t)$.)

$$x'' \overset{(\mathrm{I})}{=} x' + 2y' - \sin(t) \overset{(\mathrm{II})}{=} x' + 2y - 6x$$

$$+ 2e^{t-1} + 2t^2 - \sin(t) \overset{(\mathrm{I})}{=} x' + (x' - x - \cos(t))$$

$$- 6x + 2e^{t-1} + 2t^2 - \sin(t)$$

$$= 2x' - 7x + 2e^{t-1} + 2t^2 - \sin(t) - \cos(t)$$

$x(0) = 1$ $\qquad$ $x'(0) = x(0) + 2y(0) + \cos(0) = 1 + 8 + 1 = 10$