

MATH 33A (Lecture 2, Fall 2003)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Circle the discussion section in which you are enrolled:

2A (Tue. 10am, Stephen) 2B (Thur. 10am, Stephen)

2C (Tue. 10am, Brian) 2D (Thur. 10am, Brian)

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. In questions where there is a “yes or no” answer, the grading is always based on the explanation rather than on the answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	8
Score								

Question	9	10	11	12	13	14	15	Total
Score								

1) (10 points) Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer carefully (no credit for right answer for wrong reason).

The coefficient matrix A is 4×3 .

Unique solution \Rightarrow each column of $\text{rref}(A)$ has leading 1.

So only possibility is

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2) (10 points) Are the vectors below linearly independent? Explain your answer carefully (no credit for right answer for wrong reason).

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 7 \\ 9 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 4 \\ 5 \\ 7 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Must check if matrix A below has rank 3:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \\ 7 & 7 & 1 \\ 9 & 2 & 1 \end{bmatrix}$$

Let's find $\text{ref}(A)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \\ 7 & 7 & 1 \\ 9 & 2 & 1 \end{bmatrix} \begin{array}{l} -7(I) \\ -9(I) \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \\ 0 & 7 & 1 \\ 0 & 2 & 1 \end{bmatrix} \div 4$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 7 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{array}{l} -5(II) \\ -7(II) \\ -2(II) \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \text{swap} \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - (III)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So rank $A = 3$

Answer: Yes

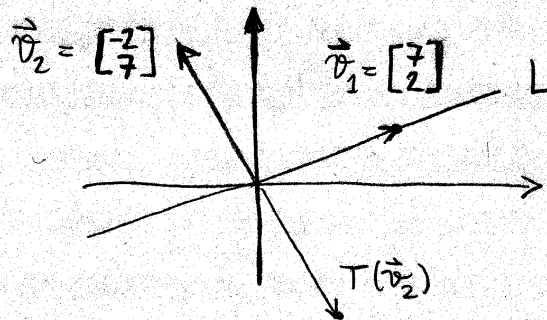
3) (10 points) Suppose that T is the linear transformation from \mathbb{R}^2 to \mathbb{R}^2 corresponding to the reflection in the line L spanned by the vector

$$\vec{v} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Find the matrix B of T with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, where

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

(Note that \vec{v}_1 and \vec{v}_2 are orthogonal to each other.)



$$[T(\vec{v}_1)]_{\mathcal{B}} = [\vec{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad [T(\vec{v}_2)]_{\mathcal{B}} = [-\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & [T(\vec{v}_2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

4) (10 points) Suppose that A and B are invertible $n \times n$ matrices. Can one conclude that $(ABA^{-1})^9 = AB^9A^{-1}$? Explain your answer carefully (no credit for right answer for wrong reason).

$$\begin{aligned}
 (ABA^{-1})^9 &= \underbrace{ABA^{-1} ABA^{-1} ABA^{-1} \dots ABA^{-1} ABA^{-1}}_{9 \text{ times}} \\
 &= A B B B B B B B B A^{-1} = A B^9 A^{-1} \\
 &\quad \uparrow \\
 &\text{cancel terms } A^{-1}A = I_n
 \end{aligned}$$

Answer: Yes

5) (10 points) Consider an $m \times n$ matrix A and an $n \times p$ matrix B . If $\ker(A) = \text{im}(B)$, what can you say about the product AB ? Explain your answer carefully (no credit for right answer for wrong reason).

Let's look at $AB\vec{z}$ for some $\vec{z} \in \mathbb{R}^p$

$$AB\vec{z} = A(B\vec{z})$$

$$B\vec{z} \in \text{im}(B) = \ker(A)$$

But this means that

$$A(B\vec{z}) = \vec{0}$$

$$\text{So } AB\vec{z} = \vec{0} \text{ for every } \vec{z} \in \mathbb{R}^p$$

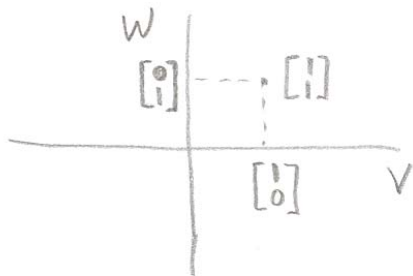
This means AB is the matrix $m \times p$ which has all entries 0.

6) (10 points) If V and W are subspaces of \mathbb{R}^2 , is their union $V \cup W$ necessarily also a subspace of \mathbb{R}^2 ? Explain your answer carefully (no credit for right answer for wrong reason).

Answer : No

For instance can have

$$V = \left\{ \begin{bmatrix} k \\ 0 \end{bmatrix} : k \in \mathbb{R} \right\} \quad W = \left\{ \begin{bmatrix} 0 \\ k \end{bmatrix} : k \in \mathbb{R} \right\}$$



But $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in W$

while $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin W$

7) (10 points) Find a basis of the subspace of \mathbb{R}^3 defined by the equation

$$x_1 + x_2 + 2x_3 = 0.$$

Can take $x_2 = s$
 $x_3 = t$ arbitrary.

$$\text{Get } x_1 = -s - 2t$$

So this subspace is $\left\{ \begin{bmatrix} -s-2t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$

$$= \left\{ s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Answer: } \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

8) (10 points) Do the 3 polynomials below form a basis of P_2 ? (Recall that P_2 is the set of polynomials of degree at most 2.) Explain your answer (no credit for right answer for wrong reasoning).

$$f(x) = 2x^2 - x + 5 \quad g(x) = x^2 + 1 \quad h(x) = 3x$$

This is the same as asking if the vectors

$$\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

form a basis of \mathbb{R}^3 . For this we check if the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 3 \\ 5 & 1 & 0 \end{bmatrix}$$

is invertible. Indeed, its determinant is

$$\det(A) = 15 - 6 = 9 \neq 0. \quad \text{So } A \text{ is invertible}$$

and the answer is yes.

10) (10 points) Find the orthogonal projection of

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

onto the subspace W of \mathbb{R}^3 spanned by \vec{x} and \vec{y} given below.

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$W = \text{span}(\vec{x}, \vec{y})$. First will find orthonormal basis of W : \vec{w}_1, \vec{w}_2

$\|\vec{x}\|=1$, so can take $\vec{w}_1 = \vec{x}$, $V_1 = \text{span}(\vec{x})$

$$\vec{y} - \text{proj}_{V_1} \vec{y} = \vec{y} - (\vec{y} \cdot \vec{w}_1) \cdot \vec{w}_1 = \vec{y} - \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \frac{\vec{y} - \text{proj}_{V_1} \vec{y}}{\|\vec{y} - \text{proj}_{V_1} \vec{y}\|} = \frac{1}{\sqrt{1}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now have

$$\begin{aligned} \text{proj}_W(\vec{u}) &= (\vec{u} \cdot \vec{w}_1) \vec{w}_1 + (\vec{u} \cdot \vec{w}_2) \vec{w}_2 = 1 \cdot \vec{w}_1 + 3 \vec{w}_2 \\ &= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \end{aligned}$$

11) (10 points) For which choices of x is the matrix A given below invertible?

$$\begin{bmatrix} 1 & 1 & x \\ 1 & x & x \\ x & x & x \end{bmatrix}$$

$$\det(A) = x^2 + x^2 + x^2 - x^3 - x^2 - x$$

$$= -x^3 + 2x^2 - x$$

$$= x(-x^2 + 2x - 1) = -x(x-1)^2$$

$$\det(A) \neq 0 \Rightarrow x \neq 0, x \neq 1$$

Answer : every $x \neq 0$ and $\neq 1$.

12) (10 points) Consider a 5×5 matrix A with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$. Suppose that $\det(A) = 7$. Compute $\det(B)$, where the matrix B is given below. (Explain carefully your computation.)

$$\begin{bmatrix} \vec{v}_3 \\ \vec{v}_2 \\ \vec{v}_1 \\ \vec{v}_4 - \vec{v}_1 \\ 2\vec{v}_5 \end{bmatrix}$$

To obtain B from A one can

- Subtract 1st row from 4th. \det does not change.
- Multiply last row by 2. \det is multiplied by 2.
- Switch rows 1 and 3. \det is multiplied by -1 .

$$\text{So } \det B = -2 \det A = -2 \times 7 = -14$$

13) (10 points) Suppose that A is a $n \times n$ matrix, and \vec{v} is an eigenvector with associated eigenvalue λ . Is \vec{v} also an eigenvector of $B = 2A^2 + I_n$ (where I_n is the $n \times n$ identity matrix)? If so, what is the corresponding eigenvalue? Explain your answer and your computations carefully.

Know $A\vec{v} = \lambda\vec{v}$

So
$$\begin{aligned} B\vec{v} &= (2A^2 + I_n)\vec{v} = 2AA\vec{v} + I_n\vec{v} \\ &= 2A\lambda\vec{v} + \vec{v} = 2\lambda A\vec{v} + \vec{v} \\ &= 2\lambda\lambda\vec{v} + \vec{v} = (2\lambda^2 + 1)\vec{v} \end{aligned}$$

Answer : Yes, with eigenvalue $2\lambda^2 + 1$

14) (10 points) Find the eigenvalues and eigenspaces of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Upper Triangular. Eigenvalue 1 (only)

$$E_1 = \ker(I_3 - A) = \ker \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{cases} -x_2 - x_3 = 0 \\ -2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = x_3 = 0 \end{cases}$$

$$\text{So } E_1 = \text{span} \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]$$

15) (10 points) Find a basis for the image of the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ corresponding to the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Find $\text{rref}(A)$:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow{-(1)}$$

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \text{rref}(A)$$

$\uparrow \qquad \qquad \uparrow$
pivot column

$$\text{Basis of } \text{im}(T) = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

9) (10 points) Provide the definition of the kernel of a linear transformation T from the linear space V to the linear space W .

$$\ker(T) = \{ \varphi \in V : T(\varphi) = 0 \}.$$