MATH 171 (Spring 2009)
Instructor: Roberto Schonmann
Midterm Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

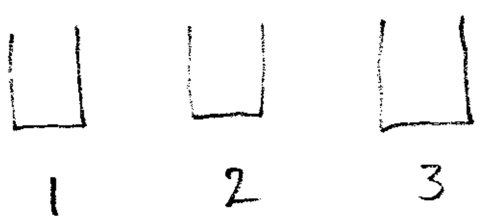
Question	1	2	3	4	5	Total
Score						

1) (10 points) Three boxes are numbered 1, 2, 3. A total of N balls is distributed among these three boxes at each time. In step n, $n = 1, 2, \ldots$ one of the balls is chosen at random and also one of the boxes is chosen at random, independently. The chosen ball is then taken and placed in the chosen box (note that if the chosen ball happens to be in the chosen box, it will not move). Let X_0 be the number of balls in box 1 at time 0, and let X_n be the number of balls in box 1, immediately after step n, $n = 1, 2, \ldots$. Find

$$p(i,j) = P(X_{n+1} = j | X_n = i),$$

$i, j = 0, 1, 2, \ldots, N$.

N balls



$p(i,j) = 0$ if $|i-j| \geq 2$ (only one ball moves)

$$p(i,i) = \frac{i}{N} \cdot \frac{1}{3} + \left(1 - \frac{i}{N}\right) \cdot \frac{2}{3} = \frac{2N - i}{3N}, \quad i = 0, 1, \ldots, N$$

$$p(i,i+1) = \left(1 - \frac{i}{N}\right) \cdot \frac{1}{3} = \frac{N-i}{3N}, \quad i = 0, 1, \ldots, N-1$$

$$p(i,i-1) = \frac{i}{N} \cdot \frac{2}{3} = \frac{2i}{3N}, \quad i = 1, 2, \ldots, N$$

2) (10 points) Consider a Markov chain with 2 states, denoted A and B. Suppose that the transition matrix p is defined by $p(A,B) = 0.1$, $p(B,A) = 0.4$. Show that, for $n = 0, 1, 2, \ldots$,

$$P_A(X_{n+1} = A) - 0.8 = 0.5 \{P_A(X_n = A) - 0.8\}.$$

$$P_A(X_{n+1} = A) = P_A(X_n = A)\, p(A,A)$$

$$+ P_A(X_n = B)\, p(B,A)$$

$$= P_A(X_n = A) \cdot 0.9 + (1 - P_A(X_n = A)) \cdot 0.4$$

$$= 0.5\, P_A(X_n = A) + 0.4$$

$$\therefore P_A(X_{n+1} = A) - 0.8 = 0.5 (P_A(X_n = A) - 0.8)$$

$$+ 0.5 \times 0.8 + 0.4 - 0.8$$

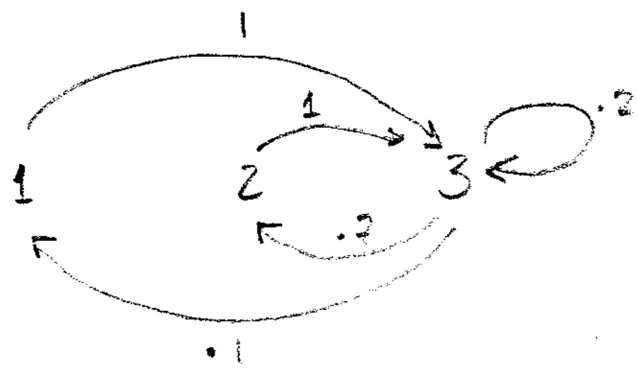
$$\therefore P_A(X_{n+1} = A) - 0.8 = 0.5 (P_A(X_n = A) - 0.8)$$

3) (10 points) A Markov chain has states 1,2,3, and transition matrix given by

$$p = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ .1 & .7 & .2 \end{bmatrix}$$

Compute.

$$\lim_{n\to\infty} p^n(1,2).$$



irreducible
aperiodic

invariant dist. π : $\pi p = \pi$ $\begin{cases} .1\pi_3 = \pi_1 \\ .7\pi_3 = \pi_2 \\ \pi_1 + \pi_2 + .2\pi_3 = \pi_3 \\ \text{(discard)} \end{cases}$

$\pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow (.1 + .7 + 1)\pi_3 = 1 \Rightarrow \pi_3 = \frac{1}{1.8} = \frac{5}{9}$

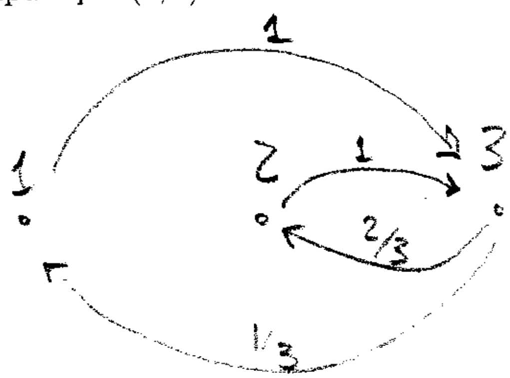
$\pi_1 = \frac{.1}{1.8} = \frac{1}{18}$, $\pi_2 = \frac{.7}{1.8} = \frac{7}{18}$

$\lim_{n\to\infty} p^n(1,2) = \pi_2 = \frac{7}{18}$

4) (10 points) A Markov chain has states 1,2,3, and transition matrix given by

$$p = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1/3 & 2/3 & 0 \end{bmatrix}$$

Compute $p^{101}(1,2)$.



Starting from state 1, the chain is at 3 on odd times and at 1 or 2 at even times

So $p^{101}(1,2) = 0$ (since 101 is odd)

5) (10 points) A Markov chain has states 1,2,3, and transition matrix given by

$$p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Does its single stationary distribution $\pi = (\pi_1, \pi_2, \pi_3)$ satisfy the detailed balance condition? (Important: show your work, a simple yes or no answer is not enough.)

Doubly stochastic matrix $\Rightarrow \pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

(irreducible, so there is only π).

Detailed balance : $\pi(x)\, p(x,y) = \pi(y)\, p(y,x)$

$x=1,\ y=2$: $\left.\begin{array}{l} \pi(x)\, p(x,y) = \frac{1}{3} \times 1 = \frac{1}{3} \\ \pi(y)\, p(y,x) = \frac{1}{3} \times 0 = 0 \end{array}\right\} \neq$

So detailed balance <u>not</u> satisfied.