MATH 171 (Spring 2009)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	8	9	10	Total
Score											

1) (10 points) Three boxes are numbered 1, 2, 3. A total of N balls is distributed among these three boxes at each time. 2 of these balls are red and the others are blue. In step n, $n = 1, 2, \ldots$ one of the balls is chosen at random and also one of the boxes is chosen at random, independently. The chosen ball is then taken and placed in the chosen box (note that if the chosen ball happens to be in the chosen box, it will not move). Let Y_0 be the number of red balls in box 1 at time 0, and let Y_n be the number of red balls in box 1, immediately after step n, $n = 1, 2, \ldots$. Find

$$p(i,j) = P(Y_{n+1} = j | Y_n = i),$$

$i, j = 0, 1, 2.$

Clearly $p(i,j) = 0$ if $|i-j| > 1$.

For $i = 0, 1$, $p(i, i+1) = \frac{2-i}{N} \cdot \frac{1}{3} = \frac{2-i}{3N}$

For $i = 1, 2$, $p(i, i-1) = \frac{i}{N} \cdot \frac{2}{3} = \frac{2i}{3N}$

For $i = 0, 1, 2$, $p(i,i) = \frac{i}{N} \cdot \frac{1}{3} + \frac{2-i}{N} \cdot \frac{2}{3} + \frac{N-2}{N}$

$$= \frac{3N - i - 2}{3N}$$

$$p = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{3N-2}{3N} & \frac{2}{3N} & 0 \\ \frac{2}{3N} & \frac{3N-3}{3N} & \frac{1}{3N} \\ 0 & \frac{4}{3N} & \frac{3N-4}{3N} \end{bmatrix} \end{array}$$

2) (10 points) Prove that if a Markov chain is irreducible and π is an invariant distribution for this chain, then for every state x we must have $\pi(x) > 0$. (Do not use any theorem; use only the definitions of irreducubility and invariance.)

Since $\sum_{y \in S} \pi(y) = 1$ and $\pi(y) \geq 0$ for each $y \in S$,

we must have $\pi(y) > 0$ for some $y \in S =$ state space.

Since the chain is irreducible, y communicates with x. This means that for some n,

$p^n(y, x) > 0$.

Since π is invariant, $\pi p = \pi$. So

$$\pi p^n = (\pi p) p^{n-1} = \pi p^{n-1} = (\pi p) p^{n-2} = \pi p^{n-2} = \dots = \pi p = \pi.$$

Therefore $$\pi(x) = \sum_{z \in S} p^n(z, x)\, \pi(z) \geq p^n(y, x)\, \pi(y)$$

$$> 0.$$

$\square$

3) (10 points) An irreducible Markov chain has a finite number N of states and has a symmetric transition matrix, i.e., $p(x, y) = p(y, x)$, for each pair of states x, y. Find its invariant distribution π.

p is doubly stochastic, since

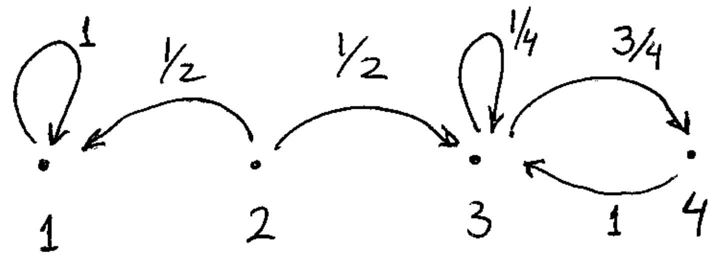
$$\sum_{x \in S} p(x,y) = \sum_{x \in S} p(y,x) = 1 .$$

Therefore $\pi = \left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N}\right)$

4) (10 points) A Markov chain has states 1,2,3,4 and transition matrix given by

$$p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Compute $\lim_{n\to\infty} p^n(x,x)$, for $x = 1,2,3,4$.



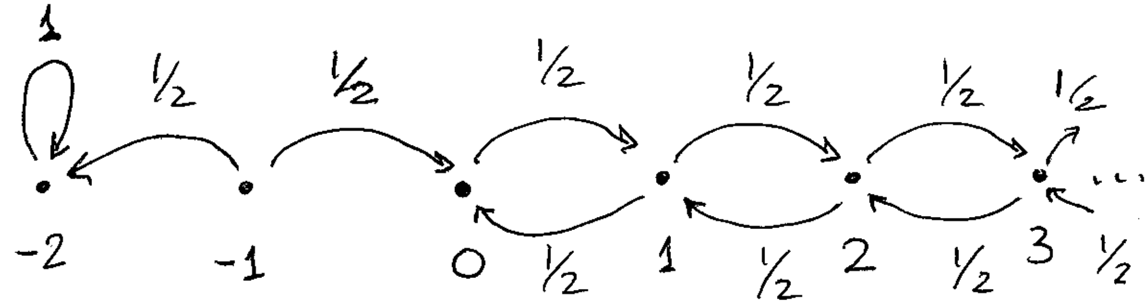
- State 1 is absorbing, so $\lim_{n\to\infty} p^n(1,1) = 1$
- State 2 is transient, since from 2 you reach 1 that is absorbing, or $\{3,4\}$ that is closed. So $\lim_{n\to\infty} p^n(2,2) = 0$
- States 3 and 4 belong to a closed irreducible set $\{3,4\}$. So we can find $\lim_{n\to\infty} p^n(x,x)$, $x = 3,4$, by considering the irreducible, aperiodic chain with

$$\tilde{p} = \begin{array}{c c} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 3 \\ 4 \end{array} & \begin{bmatrix} 1/4 & 3/4 \\ 1 & 0 \end{bmatrix} \end{array} \quad \therefore \quad \tilde{\pi}\tilde{p} = \tilde{\pi} \quad \therefore \quad \frac{3}{4}\tilde{\pi}_3 = \tilde{\pi}_4$$

and $\tilde{\pi}_3 + \tilde{\pi}_4 = 1 \quad \therefore \quad \frac{7}{4}\tilde{\pi}_3 = 1 \quad \therefore \quad \tilde{\pi}_3 = \frac{4}{7}, \quad \tilde{\pi}_4 = \frac{3}{7}. \Rightarrow \begin{cases} \lim_{n\to\infty} \tilde{p}^n(3,3) = \frac{4}{7} \\ \lim_{n\to\infty} \tilde{p}^n(4,4) = \frac{3}{7} \end{cases}$

5) (10 points) Give examples of Markov chains with the indicated properties, or explain why it is impossible to have such a chain. a) Give an example of a Markov chain with exactly one transient state, exactly one positive recurrent state and at least one null recurrent state. b) Give an example of a Markov chain with exactly one transient state, exactly one positive recurrent state and exactly one null recurrent state.

a)



$S = \{-2, -1, 0, 1, 2, 3, \dots\}$

$p(x,y)$ as in picture (with $p(x,y) = 1/2$ if $y = x+1$ or $y = x-1$, when $x = 1, 2, 3, 4, \dots$)

- State -2 is absorbing, so positive recurrent.
- State -1 is transient
- States $0, 1, 2, 3, \dots$ form an irreducible closed set of states. From gambler ruin analysis, we know that they are null recurrent

b) Impossible. Would have a finite Markov chain (3 states) and for such chains states are never null recurrent.

6) (10 points) A Markov chain has states 1,2,3,4 and transition matrix given by

$$p = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

Define $V = \min\{n \geq 0 : X_n = 4\}$, i.e., the first time that the Markov chain is in state 4 (possibly time 0). Compute $E_1(V)$, i.e., the expected value of V when the Markov chain is started from state 1.

$h(x) = E_x(V)$. $\qquad E_1(V) = h(1) = ?$

$$\begin{cases} h(1) = \frac{3}{4}(h(1)+1) + \frac{1}{4}(h(2)+1) \\ h(2) = \frac{1}{3}(h(1)+1) + \frac{1}{3}(h(3)+1) + \frac{1}{3}(h(4)+1) \\ h(3) = \frac{1}{2}(h(2)+1) + \frac{1}{2}(h(4)+1) \\ h(4) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} h(1) = h(2) + 4 \\ 3h(2) = h(1) + h(3) + \cancel{h(4)} + 3 \\ 2h(3) = h(2) + 2 \end{cases} \Rightarrow \begin{cases} h(1) = h(2) + 4 \\ \cancel{6}\,5\,h(2) = 2h(1) + \cancel{h(2)} + 8 \end{cases}$$

$$\Rightarrow \left\{ 5h(1) = 2h(1) + 8 + 20 \right. \quad \therefore \quad h(1) = \frac{28}{3} \quad \therefore \quad \boxed{E_1(V) = \frac{28}{3}}$$

7) (10 points) Suppose that X'_n and X''_n are independent Markov chains, both with two states, denoted by 1 and 2, and with the same transition matrix

$$p = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

Let $L_n = \sum_{k=0}^{n} 1_{\{X'_k = X''_k\}}$ be the number of times that the two chains are in the same state, from time 0 to time n. Compute

$$\lim_{n\to\infty} \frac{L_n}{n},$$

when initially both chains are in state 1.

p is irreducible and has stationary distribution π given by:

$$\left.\begin{array}{l} \pi p = \pi \Rightarrow \frac{1}{2}\pi_2 = \pi_1 \\ \pi_1 + \pi_2 = 1 \end{array}\right\} \Rightarrow \begin{cases} \pi_1 = 1/3 \\ \pi_2 = 2/3 \end{cases}$$

The coupled chain (X'_n, X''_n) has stationary distribution $(\tilde{\pi}_{11}, \tilde{\pi}_{12}, \tilde{\pi}_{21}, \tilde{\pi}_{22}) = (\pi_1^2, \pi_1\pi_2, \pi_2\pi_1, \pi_2^2)$

$$= (1/9, 2/9, 2/9, 4/9).$$

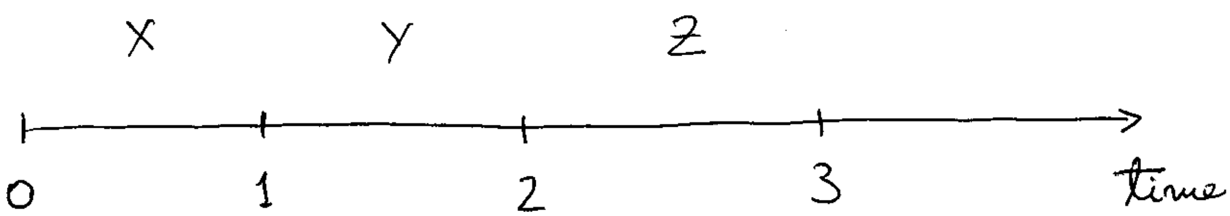
The coupled chain is irreducible. Hence, by the strong law for markov chains:

$$\lim_{n\to\infty} \frac{L_n}{n} = \tilde{\pi}_{11} + \tilde{\pi}_{22} = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}.$$

8) (10 points) Suppose that $N(t)$ is a Poisson process with rate λ. Compute the conditional probability

$$P\left(N(2) - N(1) = k \,|\, N(3) = j\right),$$

for $j = 0, 1, 2, \ldots$, and $k = 1, 2, \ldots, j$.



$X = N(1)$, $Y = N(2) - N(1)$, $Z = N(3) - N(2)$

are independent $\sim$ Poisson (λ).

$$\circledast = P\left(N(2) - N(1) = k \,|\, N(3) = j\right) = \frac{P(Y = k,\ X+Y+Z = j)}{P(X+Y+Z = j)}$$

$$= \frac{P(Y = k,\ X+Z = j-k)}{P(X+Y+Z = j)}$$

Since Y and $X+Z$ are indep., $X+Z \sim$ Poisson (2λ) and $X+Y+Z \sim$ Poisson (3λ), we have

$$\circledast = \frac{\left(e^{-\lambda}\, \lambda^k / k!\right)\left(e^{-2\lambda}\, (2\lambda)^{j-k} / (j-k)!\right)}{e^{-3\lambda}\, \frac{(3\lambda)^j}{j!}} = \binom{j}{k} \frac{2^{j-k}}{3^j}.$$

9) (10 points) Suppose that Y_n is a discrete time Markov chain with states $S = \{1, 2, 3\}$ and transition matrix

$$u = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 2/3 & 0 \\ 1/5 & 2/5 & 2/5 \end{bmatrix}$$

and suppose that $N(t)$ is an independent Poisson process with rate λ. Then we know that $X_t = Y_{N(t)}$ defines a continuous-time Markov chain, that jumps at the arrival times of the Poisson process $N(t)$, according to the transition matrix u. Let A be the event that $X_t = 3$ for $0 \le t \le 10$, i.e., the event that the continuous-time Markov chain X_t stays in the state 3 from time 0 to time 10. Compute $P_3(A)$.

$$P_3(A) = \sum_{k=0}^{\infty} \overbrace{e^{-\lambda \cdot 10} \frac{(\lambda \cdot 10)^k}{k!}}^{k \text{ jumps}} \cdot \overbrace{\left(\frac{2}{5}\right)^k}^{\text{at each jump stay in state 3}}$$

$$= \sum_{k=0}^{\infty} e^{-10\lambda} \frac{(4\lambda)^k}{k!} = e^{-10\lambda} e^{4\lambda} = e^{-6\lambda}$$

10) (10 points) Customers arrive at a full-service one-pump gas station at a rate of 15 cars per hour. However, customers will go to another station if there are two cars in the station (one being served and one waiting). Suppose that the service time for customers is exponential with mean 5 minutes. Formulate a Markov chain for the number of cars at the gas station and find its stationary distribution.

$S = \{0, 1, 2\}$ rate of service: $\frac{60}{5} = 12/\text{hour}$

$$Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} -15 & 15 & 0 \\ 12 & -27 & 15 \\ 0 & 12 & -12 \end{bmatrix} \end{array}$$

$$\pi Q = 0 \Rightarrow \begin{cases} -15\pi_0 + 12\pi_1 = 0 \\ +15\pi_1 - 12\pi_2 = 0 \end{cases} \Rightarrow \begin{cases} \pi_1 = 5/4\, \pi_0 \\ \pi_2 = 5/4\, \pi_1 \end{cases}$$

also $\pi_0 + \pi_1 + \pi_2 = 1$

$$\Rightarrow 1 = \pi_0 + \pi_1 + \pi_2 = \pi_0 \left(1 + \frac{5}{4} + \frac{25}{16}\right) = \frac{61}{16}$$

$$\Rightarrow \pi_0 = \frac{16}{61}, \quad \pi_1 = \frac{20}{61}, \quad \pi_2 = \frac{25}{61}$$