

1) (10 points) Let g be a uniformly continuous function on \mathbb{R} and $\{a_n\}$ be a numerical sequence which satisfies $a_n \rightarrow a$, for some $a \in \mathbb{R}$. Define f_n by $f_n(x) = g(x + a_n)$. Does $f_n \rightarrow f$ uniformly for some function f ? (If your answer is yes, say what the function f is.) Prove your answer.

Yes, $f_n \rightarrow f$ uniformly, where

$$f(x) = g(x + a).$$

pf: Since g is unif. cont., given $\epsilon > 0$
 $\exists \delta > 0$ s.t. $|y - z| \leq \delta \Rightarrow |g(y) - g(z)| \leq \epsilon$
(I)

Since $a_n \rightarrow a$, given $\delta > 0 \exists N$ s.t.

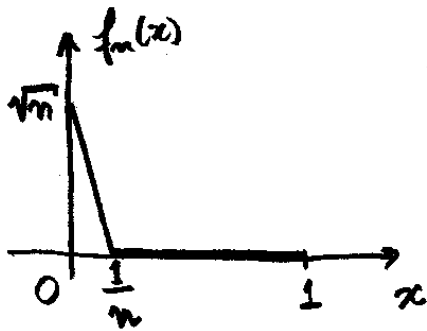
$$n \geq N \Rightarrow |a_n - a| \leq \delta \quad (\text{II})$$

From (I) and (II): $n \geq N \Rightarrow |f_n(x) - f(x)|$

$$= |g(x + a_n) - g(x + a)| \leq \epsilon$$

□

2) (10 points) Provide an example of a sequence of functions in $C[0,1]$ (continuous functions from $[0,1]$ to \mathbb{R}), which converges according to the $\|\cdot\|_1$ norm but not according to the $\|\cdot\|_\infty$ norm.



$$f_n(x) = \begin{cases} \sqrt{n} - \frac{3}{2}x & 0 \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \leq 1 \end{cases}$$

Set also $f(x) = 0 \quad 0 \leq x \leq 1$

$$\begin{aligned} f_n &\rightarrow f \text{ in } \|\cdot\|_1, \text{ since } \|f_n - f\|_1 \\ &= \int_0^1 |f_n(x) - 0| dx = \int_0^1 f_n(x) dx = \frac{\sqrt{n}}{2n} \rightarrow 0 \end{aligned}$$

But f_n does not converge to any f in $\|\cdot\|_\infty$ norm, since $\|f_n\|_\infty = \sqrt{n} \rightarrow \infty$.

(Recall that if $f_n \rightarrow g$ in $\|\cdot\|_\infty$, then $\|f_n\|_\infty \rightarrow \|g\|_\infty$.)

3) (10 points) Suppose that (M, ρ) is a metric space, $x, y \in M$, and that $\{x_n\}$ is a sequence in this metric space such that $x_n \rightarrow x$. Prove that $\rho(x_n, y) \rightarrow \rho(x, y)$.

$$\rho(x_n, y) \stackrel{\circledast}{\leq} \rho(x_n, x) + \rho(x, y)$$

$$\therefore \rho(x_n, y) - \rho(x, y) \leq \rho(x_n, x) \quad (\text{I})$$

$$\rho(x, y) \stackrel{\circledast}{\leq} \rho(x, x_n) + \rho(x_n, y)$$

$$\therefore \rho(x_n, y) - \rho(x, y) \geq -\rho(x, x_n) \quad (\text{II})$$

Since $x_n \rightarrow x$, $\rho(x, x_n) \rightarrow 0$. So (I), (II) and the squeeze theorem imply

$$\rho(x_n, y) - \rho(x, y) \rightarrow 0. \text{ Hence}$$

$$\rho(x_n, y) \rightarrow \rho(x, y)$$

\circledast triangle inequality.

4) (10 points) Is the following metric space complete? Prove your answer. $\{f \in C[0,1] \mid f(0) = 0, f(1) > 0\}$, with metric corresponding to the supremum norm $\|\cdot\|_\infty$.

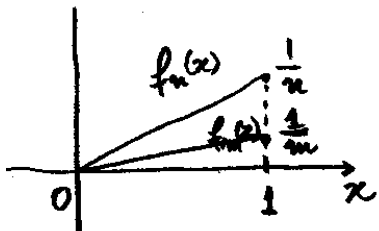
No.

Consider the functions $f_n(x) = x/n, 0 \leq x \leq 1$.

$\{f_n\}$ is a Cauchy sequence in $C[0,1]$ since

given $\varepsilon > 0$, take $N \geq 1/\varepsilon$ and then

$$\begin{aligned} m, n \geq N &\Rightarrow \|f_m - f_n\|_\infty = \left| \frac{1}{n} - \frac{1}{m} \right| \\ &\leq \frac{1}{N} \leq \varepsilon. \end{aligned}$$



But if we had $f_n \rightarrow g$ in $C[0,1]$ for some $g \in C[0,1]$ with $g(1) > 0$, then

$$|f_n(1) - g(1)| \leq \|f_n - g\|_\infty \rightarrow 0.$$

Hence $f_n(1) \rightarrow g(1)$. But actually $f_n(1) = \frac{1}{n} \rightarrow 0$.

5) (10 points) When V is a linear space with norm $\|\cdot\|$, one defines a metric on V by $\rho(v, u) = \|v - u\|$. Prove that the triangle inequality for $\rho(\cdot, \cdot)$ follows from that for $\|\cdot\|$.

$$\forall v, u, w \in V,$$

$$\rho(v, u) = \|v - u\| = \|(v - w) + (w - u)\|$$

$$\leq \|v - w\| + \|w - u\| = \rho(v, w) + \rho(w, u)$$

↑
triangle ineq.
for $\|\cdot\|$

6) (10 points) Define $\limsup a_n$.

$$\limsup a_n = \lim_{j \rightarrow \infty} \sup \{a_n \mid n \geq j\}$$

7) (10 points) Suppose that a_j is a sequence of real numbers, such that the series $\sum_{j=1}^{\infty} a_j$ converges. Define $b_{n,j} = a_{j+n}$, $n, j = 1, 2, \dots$. Show that for each n ,

$$c_n = \sum_{j=1}^{\infty} b_{n,j}$$

is well defined (i.e., this series converges), and that $c_n \rightarrow 0$ as $n \rightarrow \infty$.

Define $S_m = \sum_{j=1}^m a_j$, $S_m \rightarrow S$
 (since $\sum_{j=1}^{\infty} a_j$ converges)

Now define $S_{n,m} = \sum_{j=1}^m b_{n,j} = \sum_{j=1}^m a_{m+j}$
 $= \sum_{k=m+1}^{m+n} a_k = S_{m+n} - S_m$

Hence $\lim_{M \rightarrow \infty} S_{n,M} = S - S_n$

This means that $c_n = S - S_n$ is well defined as $\sum_{j=1}^{\infty} b_{n,j}$.

Also $\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} (S - S_n) = S - S = 0$.

8) (10 points) Prove that

$$\zeta(x) = \sum_{j=1}^{\infty} \frac{1}{j^x}$$

defines a continuous function for $x > 1$. (This is called the Riemman zeta function.)

Suppose $a > 1$. Consider first $x \in [a, \infty)$.

Then $j^x \geq j^a \therefore j^{-x} \leq j^{-a} = M_j$.

$$\sum_{j=1}^{\infty} M_j \leq \sum_{j=1}^{\infty} j^{-a} < \infty \quad (\text{since } a > 1)$$

Hence, by Weierstrass' M-test

$\sum_{j=1}^{\infty} \frac{1}{j^x}$ converges uniformly on $[a, \infty)$.

Since for each j , j^x is a continuous function of x , the limit (call it $\zeta(x)$) is continuous in $x \in [a, \infty)$. Since $a > 1$ is arbitrary these claims hold on $(1, \infty)$.

9) (10 points) Is there a power series which converges to the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

for all $x \in (-1, 1)$? Explain your answer (no partial credit for right answer for wrong reason).

No. Suppose that such a series existed, call it $\sum_{j=0}^{\infty} a_j (x-x_0)^j$. Let R be its radius of convergence. Then would need to have $(-1, 1) \subset [x_0 - R, x_0 + R]$. But the limit of the series would be a function which is C^∞ on $(x_0 - R, x_0 + R)$ and hence cannot be discontinuous (as f is) at 0 .

10) (10 points) Suppose that $f : \mathbb{N} \rightarrow \mathbb{Q}$ is a one-to-one onto function. Can the series $\sum_{j=1}^{\infty} f(j)$ converge? Prove your answer. (No credit for right answer for wrong reason.)

The series cannot converge.

Let's prove it by contradiction.

If it converged, then $f(j) \rightarrow 0$ as $j \rightarrow \infty$.

This would imply, given $\epsilon = 1 \exists J$ s.t.

$$j \geq J \Rightarrow |f(j)| \leq 1.$$

But this implies that all the rational numbers in $(-\infty, -1) \cup (1, \infty)$ are contained in $\{f(1), f(2), \dots, f(J-1)\}$.

This is absurd since an infinite set cannot be contained in a finite set.