MATH 131A (Fall 2002, Lecture 1)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	8	9	10	Total
Score											

1) (10 points) Is the set of irrational numbers countable? Explain your answer. (You can use without proof anything we proved about countable and uncountable sets.)

No, it is uncountable.

Explanation: We know that $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable. $\mathbb{R}$ is the union of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. If $\mathbb{R} \setminus \mathbb{Q}$ were countable, $\mathbb{R}$ would be the union of two countable sets, and hence would be countable. But this is false.

2) (10 points) Suppose that $\{a_n\}$ is a sequence which satisfies $a_n \to L$. Define

$$b_n = \begin{cases} 13 & \text{if } n \le 1000 \\ a_n & \text{if } n > 1000 \end{cases}$$

Show, using only the definition of convergence of sequences, that $b_n \to L$.

Given $\varepsilon > 0$ want to find N s.t.

$$n \geq N \Rightarrow |b_n - L| \leq \varepsilon. \quad (\mathrm{I})$$

Since $a_n \to L$, have $\exists N'$ s.t.

$$n \geq N' \Rightarrow |a_n - L| \leq \varepsilon. \quad (\mathrm{II})$$

Define $N = \max\{N', 1001\}$. Then

$$n \geq N \Rightarrow \left\{ \begin{array}{lcl} n \geq 1001 & \Rightarrow & b_n = a_n \\ n \geq N' & \underset{(\mathrm{II})}{\Rightarrow} & |a_n - L| \leq \varepsilon \end{array} \right\} \Rightarrow$$

$$\Rightarrow |b_n - L| \leq \varepsilon. \quad (\text{which is } (\mathrm{I})).$$

3) (10 points) Define bounded sequence.

$\{a_n\}$ is bounded if $\exists M$ s.t. $\forall n$

$|a_n| \leq M$.

4) (10 points) Does the sequence $a_n = 1/n$ satisfy the statement below?
For any $\epsilon > 0$ and any $N \in I\!N$

$$|a_n - a_m| \leq \epsilon \text{ whenever } n, m \geq N.$$

Explain your answer.

No.

Take $\varepsilon = 0.1$ and $N = 1$. Then

$n = 1$ and $m = 2$ satisfy $n, m \geq N$.

But $|a_n - a_m| = \left|\frac{1}{1} - \frac{1}{2}\right| = \frac{1}{2} > \varepsilon$.

5) (10 points) Give an example of a sequence that has exactly one limit point but does not converge.

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

limit point: 0

but the subsequence $\{a_{2k}\}$ has

$a_{2k} = 2k \to \infty$. So a_n does not converge.

6) (10 points) Is the function $f(x) = 3x + 2$, with domain $I\!R$, uniformly continuous? Prove your answer.

Yes. Given $\varepsilon > 0$ take $\delta = \varepsilon/3$. Then

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)|$$

$$= |(3x+2) - (3y+2)| = 3|x-y|$$

$$\leq 3\delta = \varepsilon.$$

7) (10 points) What is the negation of the following statement? For any $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

For some $\varepsilon > 0$ there is no $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$

Alternative answer:

For some $\varepsilon > 0$ for all $\delta > 0$ there are x, y s.t. $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

8) (10 points) Give an example of a function defined on $[0,1]$ that is not Riemann integrable, and explain how one can prove that it is not Riemann integrable.

Example: $f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 1 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$

For this function, for any partition P of $[0,1]$:

$L_P(f) = 0$, $U_P(f) = 1$

(because in any subinterval $[x_{i-1}, x_i]$ of the partition there are rational and irrational numbers, so $m_i = 0$, $M_i = 1$)

Therefore

$$\sup_P L_P(f) = 0 \neq 1 = \inf_P U_P(f).$$

9) (10 points) Prove that if f is differentiable at x, then f is continuous at x.

$$\lim_{h \to 0} f(x+h) = \lim_{h \to 0} \left(f(x+h) - f(x) + f(x) \right)$$

$$= f(x) + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot h$$

$$= f(x) + f'(x) \cdot 0 = f(x). \quad \square$$

10) (10 points) State the Mean Value Theorem. (You do not have to prove it.)

Suppose f continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$