

MATH 115A (Spring 2006, Lecture 2)
Instructor: Roberto Schonmann
Second Midterm Exam

Last Name:

First and Middle Names:

Solutions

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	Total
Score						

1) (10 points) Find the nullity and the rank of the linear transformation $T: P_3(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ given by $T(f(x)) = f''(x)$.

$$P_3(\mathbb{R}) = \{ a_3 x^3 + a_2 x^2 + a_1 x + a_0 : a_3, a_2, a_1, a_0 \in \mathbb{R} \}$$

$$T(a_3 x^3 + a_2 x^2 + a_1 x + a_0) = 6a_3 x + 2a_2$$

$$\text{Kernel: } N(T) = \{ f \in P_3(\mathbb{R}) : T(f) = 0 \} = \{ a_1 x + a_0 : a_1, a_0 \in \mathbb{R} \}$$

$$\text{nullity} = \dim N(T) = \underline{\underline{2}}$$

$$\text{dimension theorem: } \text{nullity} + \text{rank} = \dim(V) = \dim(P_3(\mathbb{R}))$$

$$\therefore \text{rank} = \dim(P_3(\mathbb{R})) - \text{nullity} = 4 - 2 = \underline{\underline{2}}$$

2) (10 points) Let $V = P(\mathbb{R})$, the vector space of polynomials with coefficients in \mathbb{R} . Define the linear operators $T, U \in \mathcal{L}(V)$ by $T(f(x)) = f'(x)$ and $U(f(x)) = f''(x)$. Is the set $\{T, U\}$ a linearly independent subset of $\mathcal{L}(V)$? Prove your answer.

Want to check if $aT + bU = 0 \Rightarrow a = b = 0$

$aT + bU = 0$ means $\forall f \in P(\mathbb{R}), aTf + bUf = 0$

$$\therefore af' + bf'' = 0$$

Take $f(x) = x^2$. Then $af'(x) + bf''(x) = 2ax + 2b$

But $2ax + 2b = 0 \quad \forall x \Rightarrow a = b = 0$

So the answer is yes.

3) (10 points) Let V , W and Z be vector spaces over the same field F , and let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear. Prove that $UT : V \rightarrow Z$ is linear.

$$\forall a \in F, x, y \in V$$

$$UT(ax+y) = \underset{\substack{\uparrow \\ \text{def. } UT}}{U}(T(ax+y)) = \underset{\substack{\uparrow \\ T \text{ linear}}}{U}(aT(x) + T(y))$$

$$= \underset{\substack{\uparrow \\ U \text{ linear}}}{a} U(T(x)) + U(T(y)) = \underset{\substack{\uparrow \\ \text{def } UT}}{a} UT(x) + UT(y).$$

4) (10 points) Define isomorphic vector spaces.

The vector spaces V and W are isomorphic if there exists a linear transformation $T: V \rightarrow W$ that is invertible.

5) (10 points) Let $A \in M_{n \times n}$ and let \mathcal{S} be the set of matrices in $M_{n \times n}$ that are similar to A . Is \mathcal{S} a subspace of $M_{n \times n}$? Prove your answer.

Need $0 \in \mathcal{S}$. Let's check it:

$$0 = Q^{-1} A Q \quad \text{for some invertible } Q \in M_{n \times n}.$$

$$\text{Then } Q 0 Q^{-1} = A \quad \therefore A = 0.$$

So, unless $A=0$, answer is No.

$$\begin{aligned} \text{Case } \underline{A=0}: \quad \mathcal{S} &= \{ Q^{-1} 0 Q : Q \text{ invertible in } M_{n \times n} \} \\ &= \{ 0 \} \quad \text{which is a } \underline{\underline{\text{subspace}}} \end{aligned}$$