

MATH 115A (Spring 2006, Lecture 2)

Instructor: Roberto Schonmann

First Midterm Exam

Last Name:

First and Middle Names:

*Solutions*

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

**Good Luck !**

Question	1	2	3	4	5	Total
Score						

1) (10 points) Using only the axioms which define vector spaces, prove the cancellation law:  $\forall x, y, z \in V$ , if  $x + z = y + z$ , then  $x = y$ . State clearly which axioms you use.

By (VS4)  $\exists v \in V$  s.t.  $z + v = 0$ . (I)

$$x \stackrel{(VS3)}{=} x + 0 \stackrel{(I)}{=} x + (z + v) \stackrel{(VS2)}{=} (x + z) + v$$

$$\stackrel{\text{hypothesis}}{=} (y + z) + v \stackrel{(VS2)}{=} y + (z + v) \stackrel{(I)}{=} y + 0 \stackrel{(VS3)}{=} y$$

□

2) (10 points) Is  $W = \{(a_1, \dots, a_n) \in F^n \mid a_1 + \dots + a_n = 1\}$  a subspace of  $F^n$ ? Prove your answer.

Not a subspace since  $0 = (0, 0, \dots, 0) \notin W$ ,

because  $0 + 0 + \dots + 0 = 0 \neq 1$ .

3) (10 points) Define the meaning of "S is a linearly independent set".

S is linearly independent if for every  
 $v_1, v_2, \dots, v_n \in S$  and  $a_1, a_2, \dots, a_n \in F$ ,  
the equation  $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$   
can only be true if  $a_1 = a_2 = \dots = a_n = 0$ .

4) (10 points) Suppose that  $V$  is a vector space over  $\mathbb{R}$  and that  $\{v, u\} \subset V$  generates  $V$ . Does it follow necessarily that also the set  $\{v+u, v-u\}$  generates  $V$ ? Prove your answer.

Given  $x \in V$ , want to write

$$x = a(v+u) + b(v-u), \text{ for some } a, b \in \mathbb{R}.$$

This is the same as

$$x = (a+b)v + (a-b)u.$$

Since  $\{v, u\}$  generates  $V$ ,  $\exists r, s \in \mathbb{R}$  s.t.

$$x = rv + su.$$

So we need to find  $a, b$  s.t. 
$$\begin{cases} a+b=r \\ a-b=s \end{cases}$$

This is easy:  $a = \frac{r+s}{2}$ ,  $b = \frac{r-s}{2}$ .

So  $\{v+u, v-u\}$  generates  $V$ .

5) (10 points) Find the dimension of the subspace of  $\mathcal{F}(\mathbb{R}, \mathbb{R})$  generated by the functions  $f(x) = e^x$ ,  $g(x) = x^2 e^x$  and  $h(x) = e^x(1 + x^2)$ . Prove your answer.

Call  $\text{span}\{f, g, h\} = W$

Clearly  $h = f + g$ . So also  $\text{span}\{f, g\} = W$

Let's check that  $\{f, g\}$  l.i. :

$$af + bg = 0 \Rightarrow \forall x \in \mathbb{R} \quad ae^x + bx^2e^x = 0$$

$$\left\{ \begin{array}{l} \text{for } x=0 \Rightarrow a + b \cdot 0 = 0 \Rightarrow a = 0 \\ \text{for } x=1 \Rightarrow ae + be = 0 \Rightarrow b = -a = 0 \end{array} \right.$$

So  $\{f, g\}$  l.i. . Since it also generates  $W$ ,

$\{f, g\}$  is basis for  $W$ .

$$\text{So } \dim(W) = 2$$

Axioms which define a vector space  $V$  over a field  $F$ :

$$(VS 1) \forall x, y \in V, x + y = y + x.$$

$$(VS 2) \forall x, y, z \in V, (x + y) + z = x + (y + z).$$

$$(VS 3) \exists 0 \in V \text{ s. t. } \forall x \in V, x + 0 = x.$$

$$(VS 4) \forall x \in V, \exists y \in V \text{ s. t. } x + y = 0.$$

$$(VS 5) \forall x \in V, 1x = x.$$

$$(VS 6) \forall a, b \in F, \forall x \in V, (ab)x = a(bx).$$

$$(VS 7) \forall a \in F, \forall x, y \in V, a(x + y) = ax + ay.$$

$$(VS 8) \forall a, b \in F, \forall x \in V, (a + b)x = ax + bx.$$