MATH 115A (Spring 2006, Lecture 2)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names: Solutions

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	8	Total
Score									

1) (10 points) Suppose that V is a vector space over $I\!R$ and that $\{v, u, w\} \subset V$ is linearly independent. Does it follow necessarily that also the set $\{v+u, v-u, w\}$ is linearly independent? Prove your answer.

Suppose $a(v+u) + b(v-u) + cw = 0$

Then $(a+b)v + (a-b)u + cw = 0$

Since $\{v, u, w\}$ l.i., must have

$$\begin{cases} a+b = 0 \\ a-b = 0 \\ c = 0 \end{cases}$$

Hence $a = b = c = 0$

2) (10 points) Prove that if T is a linear transformation from the vector space V to the vector space W, then $T(0_V) = 0_W$, where 0_V and 0_W are the zero vectors of V and W, respectively.

$$T(0_V) = T(0 \cdot 0_V) = 0 \cdot T(0_V) = 0_W .$$

3) (10 points) Let $P(\mathbb{R})$ be the vector space of polynomials of arbitrary degree, with real coefficients. Consider the linear transformation $T : P(\mathbb{R}) \to P(\mathbb{R})$, given by $T(f) = f'$. Is T one-to-one? Is T onto? Prove your answers.

Not one-to-one:

$\forall f \in P(\mathbb{R})$ of degree 0, $f(x) = c$,

have $Tf = 0$.

Onto: $\forall f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$,

have $f(x) = T(g(x))$, where

$$g(x) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \dots + a_0 x$$

4) (10 points) Suppose that V and W are finite dimensional vector spaces over the same field F, $\{v_1, v_2, ..., v_n\}$ is a basis for V, and $T : V \to W$ is an isomorphism. Prove that $\{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for W.

It is sufficient to check that $\{T(v_1), \dots, T(v_n)\}$ is l.i., since a set of $n = \dim W = \dim V$ vectors in W that is l.i. is a basis for W.

Suppose $a_1 T(v_1) + \dots + a_n T(v_n) = 0$.

Then $T(a_1 v_1 + \dots + a_n v_n) = 0$ (T linear)

Since T is one-to-one, $N(T) = \{0_V\}$.

So $a_1 v_1 + \dots + a_n v_n = 0_V$.

Since $\{v_1, \dots, v_n\}$ basis for V,

$a_1 = \dots = a_n = 0$.

5) (10 points) One of the first theorems that we learned when we introduced eigenvectors, is the following.

Theorem: *A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T.*

Prove the "only if" part of this theorem.

Have $\exists\, \beta = \{v_1, \ldots, v_n\}$ s.t.

$$[T]_\beta = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}.$$

$$[Tv_j]_\beta = [T]_\beta [v_j]_\beta = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ a_j \\ \vdots \\ 0 \end{bmatrix}$$

So $Tv_j = a_j v_j$. This means that v_j is eigenvector of T.

6) (10 points) Is the the matrix A below diagonalizable in $M_{3\times 3}(I\!R)$? Explain what you do.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\det(A - \lambda I_3) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix} = \lambda^2(3-\lambda)$$

Eigenvalues: $\begin{cases} \lambda_1 = 0, & m_1 = 2 \\ \lambda_2 = 3, & m_2 = 1 \end{cases}$

Enough to check that E_{λ_1} has dimension 2:

$$E_{\lambda_1} = N(A - 0I_3) = N(A) = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : \begin{bmatrix} c \\ 2c \\ 3c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

So $\dim E_{\lambda_1} = 2$.

Answer: Yes

7) (10 points) The following theorem was proved in the course.

Theorem: *Let T be a linear operator on a vector space V and $v_1, ..., v_k$ be eigenvectors corresponding, respectively, to the distinct eigenvalues $\lambda_1, ..., \lambda_k$. Then $\{v_1, ..., v_k\}$ is linearly independent.*

The proof can be done by induction. Clearly $\{v_1\}$ is linearly independent, since $v_1 \neq 0$. So, complete the proof of that theorem, by showing that if $\{v_1, ..., v_{k-1}\}$ is linearly independent, then under the hypothesis of the theorem it will follow that $\{v_1, ..., v_k\}$ is linearly independent.

Suppose $a_1 v_1 + \dots + a_{k-1} v_{k-1} + a_k v_k = 0 \quad$ (I)

Apply $(T - \lambda_k I_V)$ to both sides:

$a_1(\lambda_1 v_1 - \lambda_k v_1) + \dots + a_{k-1}(\lambda_{k-1} v_{k-1} - \lambda_k v_k) + 0 = 0$

$\therefore a_1(\lambda_1 - \lambda_k) v_1 + \dots + a_{k-1}(\lambda_{k-1} - \lambda_k) v_k = 0.$

Since $\{v_1, \dots, v_{k-1}\}$ are assumed l. i.,

must have $a_1(\lambda_1 - \lambda_k) = \dots = a_{k-1}(\lambda_{k-1} - \lambda_k) = 0.$

Since λ_k distinct from $\lambda_1, \dots, \lambda_{k-1}$:

$a_1 = \dots = a_{k-1} = 0. \quad$ (II)

Using (II) and (I): $a_k v_k = 0 \quad \therefore \quad a_k = 0$, (III)

since $v_k \neq 0$. (II), (III) $\Rightarrow a_1 = \dots = a_k = 0$.

8) (10 points) Let V be an inner product vector space. Given $S \subset V$, define the orthogonal complement if S, denoted by $S^{\perp}$.

$$S^{\perp} = \{ y \in V : \langle x, y \rangle = 0 \ \forall x \in S \}.$$