MATH 115A (Spring 2006, Lecture 1)
Instructor: Roberto Schonmann
Final Exam

Last Name: Solutions

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	8	Total
Score									

1) (10 points) Suppose that V is a vector space over $\mathbb{R}$ and that $\{v, u, w\} \subset V$ generates V. Does it follow necessarily that also the set $\{v+u, v-u, w\}$ generates V? Prove your answer.

Given $x \in V$ want to find $a, b, c \in \mathbb{R}$

s.t. $a(v+u) + b(v-u) + cw = x$

Same as $(a+b)v + (a-b)u + cw = x$.

Since $\{v, u, w\}$ generates V, $\exists\, r, s, t \in \mathbb{R}$

s.t. $rv + su + tw = x$.

Need $\begin{cases} a+b = r \\ a-b = s \\ c = t \end{cases}$

Solved by $\begin{cases} a = \frac{r+s}{2} \\ b = \frac{r-s}{2} \\ c = t \end{cases}$

Answer: Yes

2) (10 points) Prove that if T is a linear transformation from the vector space V to the vector space W, then $T(0_V) = 0_W$, where 0_V and 0_W are the zero vectors of V and W, respectively.

$$T(0_V) = T(0 \cdot 0_V) = 0 \cdot T(0_V) = 0_W .$$

3) (10 points) Say whether the following statement is always true or not, and prove your answer. Suppose that V and W are finite dimensional vector spaces over the same field F. Then, given arbitrary vectors $v_1, v_2 \in V$ and $w_1, w_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$.

No. If $v_2 = a v_1$, then

$w_2 = T(v_2) = T(a v_1) = a T(v_1) = a w_1$.

An explicit counterexample: $v_1 \neq 0$, $v_2 = 0$

$w_1 = w_2 = 0$. Cannot have $T(v_1) = w_1$.

4) (10 points) Consider the linear transformation $T : P_3(\mathbb{R}) \to \mathbb{R}^4$ given by

$$T(ax^3 + bx^2 + cx + d) = (b + c + d,\, b + c,\, c + d,\, b + c).$$

Is T an isomorphism? Prove your answer.

isomorphism : one-to-one, onto,

But T is not one-to-one, since

$T(ax^3) = 0 \quad \forall a \in \mathbb{R}$.

Answer: No

5) (10 points) One of the first theorems that we learned when we introduced eigenvectors, is the following.

Theorem: *A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T.*

Prove the "if" part of this theorem.

Want to prove $\exists\, \beta = \{v_1, \dots, v_n\}$, $T(v_i) = \lambda_i v_i$, $i = 1, \dots, n$

$\Rightarrow$ T diagonalizable.

Pf: $[T]_\beta = A$, where A_{ij} is given by $T(v_j) = \sum_{i=1}^{n} A_{ij} v_i$.

$$\Rightarrow \lambda_j v_j = \sum_{i=1}^{n} A_{ij} v_i \Rightarrow A_{ij} = \begin{cases} \lambda_j & (i = j) \\ 0 & (i \neq j) \end{cases}$$

$\Rightarrow$ A is diagonal matrix.

6) (10 points) Is the the matrix A below diagonalizable in $M_{3\times 3}(I\!R)$? Explain what you do.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\det(A - \lambda I_3) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (-\lambda)^2 \cdot (3-\lambda)$$

Eigenvalues : $\lambda_1 = 0$ ..... multiplicity $2 = m_1$

$\lambda_2 = 3$ ... multiplicity $1 = m_2$

Eigenspace of $\lambda_1 = E_0 = N(A - 0I_3) = N(A)$

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} b = 0 \\ 2c = 0 \\ 3c = 0 \end{cases} \iff \begin{cases} a \in \mathbb{R} \text{ arbitrary} \\ b = 0 \\ c = 0 \end{cases}$$

So $\dim E_0 = 1 \neq m_1$

$\Longrightarrow$ A not diagonalizable.

7) (10 points) Define eigenspaces.

Given a vector space V and $T \in \mathcal{L}(V)$ and an eigenvalue λ of T, we define $E_\lambda = \{ v \in V : T(v) = \lambda v \}$.

8) (10 points) Let V be an inner product vector space, $x, y \in V$. Prove the triangle inequality $||x+y|| \leq ||x|| + ||y||$. (You can use, without having to prove it, the Cauchy-Schwarz inequality: $|\langle x, y\rangle| \leq ||x|| \cdot ||y||$.)

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x+y \rangle + \langle y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \overline{\langle x, y \rangle} + \langle y, y \rangle$$

$$= \|x\|^2 + 2\,\mathrm{Re}(\langle x, y \rangle) + \|y\|^2$$

$$\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

Cauchy-Schwarz

$$\therefore \|x+y\| \leq \|x\| + \|y\| .$$