MATH 113 (Winter 2005, Lecture 2)
Instructor: Roberto Schonmann
Midterm Exam

Last Name:

First and Middle Names: Solutions

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	Total
Score						

1) (10 points) A group has 3 women and 7 men. In how many ways can a committee of 4 with equal number of women and men be chosen? (You should provide a numerical answer.)

choice of women

choice of men

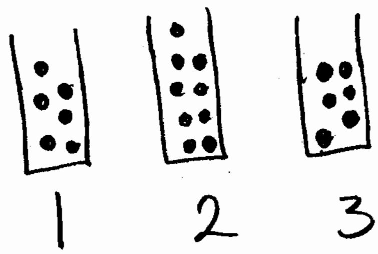
$$\# \text{ways} = \binom{3}{2} \times \binom{7}{2} = \frac{3\times 2}{2} \times \frac{7\times 6}{2}$$

$$= 3 \times 7 \times 3 = \boxed{63}$$

2) (10 points) In how many ways can we write the number 21 as the sum of three positive integer numbers (i.e., numbers 1, 2, 3, ...) if the order in which the sum is written matters (for instance, 2 + 4 + 15 is counted separately from 4 + 2 + 15). (You should provide a numerical answer.)

Same as placing 21 indistinguishable balls into 3 distinguishable cells (cells 1, 2, 3). Think of number of balls in cell i as the i-th number in the addition:

e.g.



$\longleftrightarrow \; 6+9+6$

Each cell must have at least one ball. Put one ball in each cell.

Remaining number of balls: $21-3=18$

$$\# \text{ ways} = \binom{18+2}{2} = \binom{20}{2} = \frac{20 \times 19}{2} = \boxed{190}$$

3) (10 points) Recall that $S(n,k)$ is the number of ways to place n distinguishable balls in k indistinguishable cells, with no cell empty. Use the notation $T(n,k)$ for the number of ways to place n distinguishable balls in k distinguishable cells, with no cell empty. What is the mathematical relationship between $S(n,k)$ and $T(n,k)$, and why does it hold?

$$S(n,k) \cdot k! = T(n,k)$$

Once balls are in indistinguishable cells, we can label the k cells as 1, 2, 3, ..., k. There are k! ways to do it, and this gives the ways to place the balls in distinguishable cells (distinguished by labels). The multiplication rule gives then the relationship above.

4) (10 points) In a box you have 4 black balls, 5 red balls and 3 green balls. You want to take k balls from this box, with the condition that the number of black and the number of red balls be odd and the number of green balls be even. Disregarding the order in which the balls are taken, let a_k be the number of ways in which you can do it. Compute a_k for $k = 1, 2, 3, \ldots$. ($a_0 = 1$ by convention.)

$$G(x) = \underbrace{(x + x^3)}_{\text{black}} \cdot \underbrace{(x + x^3 + x^5)}_{\text{red}} \cdot \underbrace{(1 + x^2)}_{\text{green}}$$

$$= (x^2 + 2x^4 + 2x^6 + x^8) \cdot (1 + x^2)$$

$$= x^2 + 3x^4 + 4x^6 + 3x^8 + x^{10}$$

$\Rightarrow$

$a_1 = 0$	$a_6 = 4$	$a_k = 0$ if $k \geq 11$
$a_2 = 1$	$a_7 = 0$	
$a_3 = 0$	$a_8 = 3$	
$a_4 = 3$	$a_9 = 0$	
$a_5 = 0$	$a_{10} = 1$	

5) (10 points) Find a_n if $a_0 = 0$, $a_1 = 1$ and

$$a_n = 3a_{n-1} - 2a_{n-2}.$$

Try $a_n = x^n \Rightarrow$ Ch. eq. $x^2 - 3x + 2 = 0$

$$\Rightarrow x = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$$

$$a_n = \lambda_1 1^n + \lambda_2 2^n = \lambda_1 + \lambda_2 2^n$$

$$\begin{cases} a_0 = 0 \Rightarrow \lambda_1 + \lambda_2 = 0 \\ a_1 = 1 \Rightarrow \lambda_1 + 2\lambda_2 = 1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 1 \end{cases}$$

$$\Rightarrow \boxed{a_n = -1 + 2^n}$$