**MATH 113 (Winter 2005, Lecture 1)**
Instructor: Roberto Schonmann
**Midterm Exam**

**Last Name:**

**First and Middle Names:** Solutions

**Signature:**

**UCLA id number (if you are an extension student, say so):**

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

**Good Luck !**

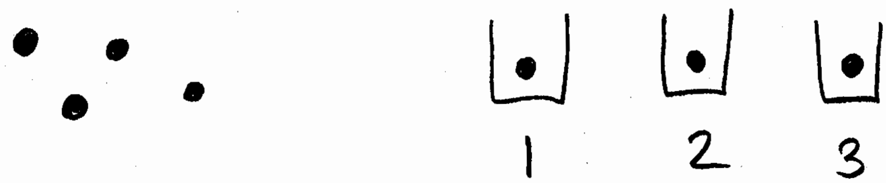
| Question | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| Score | | | | | | |

1) (10 points) In how many ways can 7 indistinguishable items be divided into 3 distinguishable groups if there must be at least one item in each group. (You should provide a numerical answer.)

items = balls

groups = cells


First place one ball in each cell




Now divide remaining 4 balls into 3 cells: $\bullet \mid \bullet\ \bullet \mid \bullet$ , etc

$$\# \text{ ways} = \binom{4+3-1}{3-1} = \binom{6}{2} = \frac{6\times5}{2} = \boxed{15}$$

2) (10 points) You have 6 books. Of these 3 have yellow cover, 2 have blue cover and 1 has brown cover. In how many ways can you place these books in order on a shelf, if those with cover of the same color must stay next to each other? (You should provide a numerical answer.)

1) Decide order of colors $\rightarrow$ $3! = 6$ ways

2) Decide order of yellow books $\rightarrow$ $3! = 6$ ways

3) Decide order of blue books $\rightarrow$ $2! = 2$ ways

4) (Nothing to decide about brown book)

Multiplication rule $\Rightarrow$ #ways $= 6 \times 6 \times 2$

$= \boxed{72}$

3) (10 points) Recall that $S(n,k)$ is the number of ways to place $n$ distinguishable balls in $k$ indistinguishable cells, with no cell empty. Use the notation $T(n,k)$ for the number of ways to place $n$ distinguishable balls in $k$ distinguishable cells, with no cell empty. What is the mathematical relationship between $S(n,k)$ and $T(n,k)$, and why does it hold?

$$S(n,k) \cdot k! = T(n,k)$$

Once balls are in indistinguishable cells, we can label the k cells as 1, 2, 3, ..., k. There are k! ways to do it, and this gives the ways to place the balls in distinguishable cells (distinguished by labels). The multiplication rule gives then the relationship above.

4) (10 points) The ordinary generating function of a sequence $(a_k)$ is given by

$$G(x) = \frac{2+3x^2}{(1+x)^3}.$$

Compute $a_5$. (You should provide a numerical answer.)

$G(x) = C(x) \cdot B(x)$

$C(x) = (2+3x^2) \implies (c_k) = (2, 0, 3, 0, 0, 0, \ldots)$

$B(x) = (1+x)^{-3} \implies b_k = \binom{-3}{k}$ (binomial thm.)

$(a_k) = (b_k) * (c_k)$

$\implies a_5 = \cancel{b_0 c_5} + \cancel{b_1 c_4} + \cancel{b_2 c_3} + b_3 c_2 + \cancel{b_4 c_1}$

$+ b_5 c_0 = \binom{-3}{3} \times 3 + \binom{-3}{5} \times 2$

$= \dfrac{\cancel{(-3)} \times (-\cancel{4})^{2} \times (-5)}{\cancel{3} \times \cancel{2}} \times 3$

$+ \dfrac{(-3) \times (-\cancel{4}) \times (-\cancel{5}) \times (-\cancel{6}) \times (-7)}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}} \times 3$

$= -30 - 42 = \boxed{-72}$

5) (10 points) Find $a_n$ if $a_0 = a_1 = 1$ and

$$a_n = 3a_{n-1} - 2a_{n-2}.$$

Try $a_n = x^n$

characteristic eq. $x^2 - 3x + 2 = 0$

$$\Rightarrow \quad x = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \begin{matrix} \nearrow 2 \\ \searrow 1 \end{matrix}$$

$$a_n = \lambda_1 2^n + \lambda_2 1^n = \lambda_1 2^n + \lambda_2$$

$$\begin{cases} a_0 = 1 \Rightarrow \lambda_1 + \lambda_2 = 1 \\ a_1 = 1 \Rightarrow 2\lambda_1 + \lambda_2 = 1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 1 \end{cases}$$

$$\Rightarrow \boxed{a_n = 1}$$