

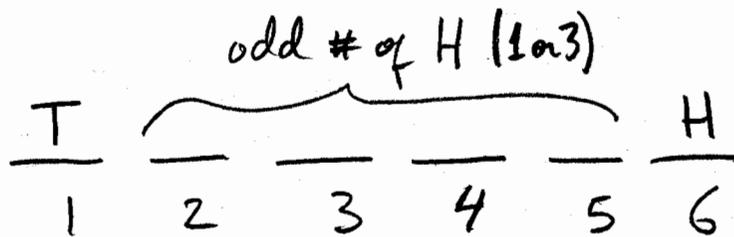
1) (10 points) In how many ways can one place 5 indistinguishable balls into 3 distinguishable cells? (You should provide a numerical answer.)



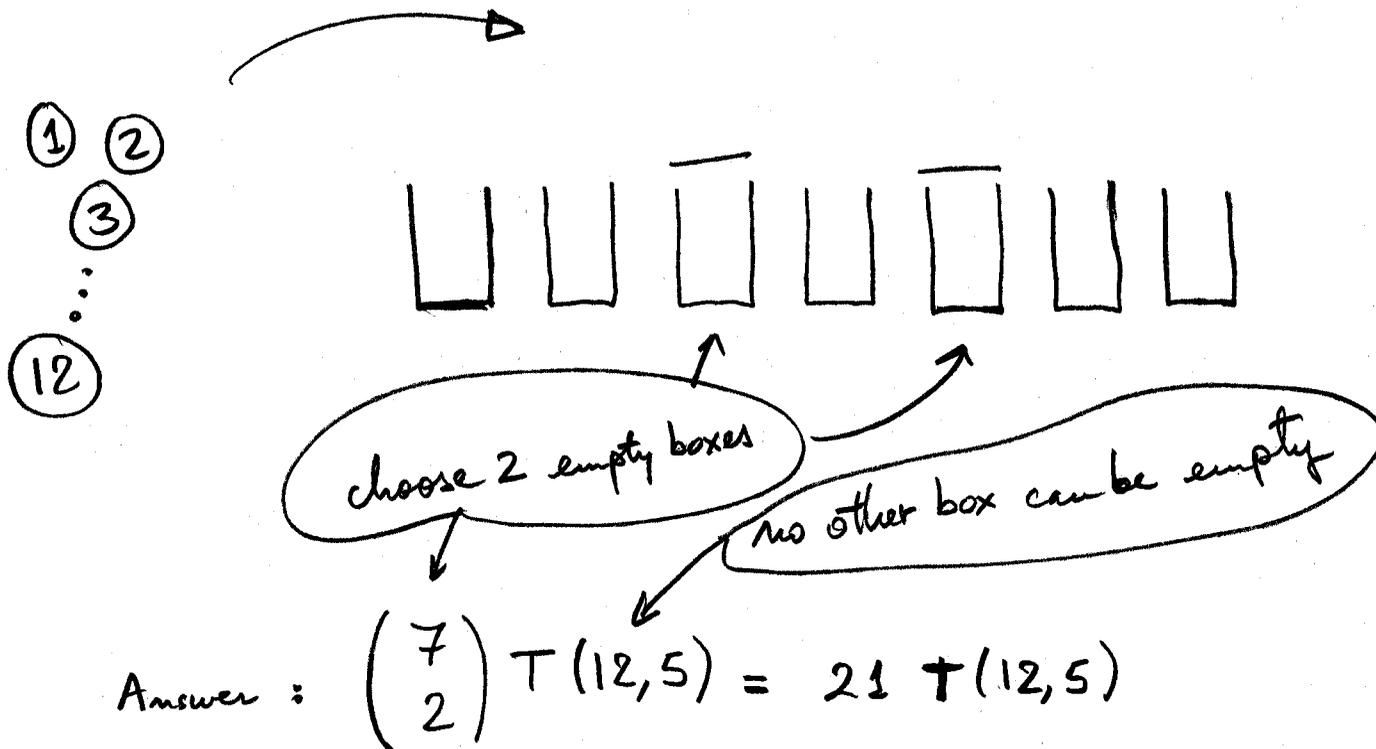
$$\text{Answer: } \binom{7}{2} = \frac{7 \times 6}{2} = \boxed{21}$$

2) (10 points) A fair coin is tossed 6 times. Compute the probability of getting an even total number of heads, a tail on the first toss and a head on the last toss. (You should provide a numerical answer.)

$$\text{Probability} = \frac{\binom{4}{1} + \binom{4}{3}}{2^6} = \frac{4+4}{2^6} = \frac{1}{8} = \boxed{.125}$$



3) (10 points) Use the notation $T(n, k)$ for the number of ways to place n distinguishable balls in k distinguishable cells, with no cell empty. In how many ways can one place 12 distinguishable balls in 7 distinguishable cells, with exactly 2 cells empty. Your answer should be given in terms of $T(n, k)$ for some appropriate value of n and k .



4) (10 points) How many codewords of length k can be written using the letters A, B, C, with A only allowed an even number of times and B only allowed an odd number of times?

Use exponential generating function (order relevant)

$$H(x) = \underbrace{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right)}_A \underbrace{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)}_B$$

$$\cdot \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)}_C = \left(\frac{e^x + e^{-x}}{2}\right) \left(\frac{e^x - e^{-x}}{2}\right) e^x$$

$$= \frac{1}{4} \left(e^{2x} - e^{-2x}\right) e^x = \frac{e^{3x} - e^{-x}}{4}$$

$$= \sum_{k=0}^{\infty} \frac{1}{4} \left\{ \frac{(3x)^k}{k!} - \frac{(-x)^k}{k!} \right\} = \sum_{k=0}^{\infty} \frac{3^k - (-1)^k}{4} \frac{x^k}{k!}$$

Answer: $\frac{3^k - (-1)^k}{4}$

5) (10 points) Suppose that the sequence (a_k) satisfies

$$a_k = 3a_{k-1} + 4a_{k-2}, \quad a_0 = 1, \quad a_1 = 2.$$

Compute

$$\lim_{k \rightarrow \infty} \frac{a_k}{a_{k-1}}.$$

Solve recursion: $a_k = x^k$ Ch. eq. $x^2 - 3x + 4 = 0$

$$\Rightarrow x = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = \begin{matrix} \nearrow 4 \\ \searrow -1 \end{matrix}$$

General solution: $a_k = A4^k + B(-1)^k$, $A, B \in \mathbb{R}$

Can compute A, B from $a_0=1, a_1=2$, but no need.

$$\lim_{k \rightarrow \infty} \frac{a_k}{a_{k-1}} = \lim_{k \rightarrow \infty} \frac{A4^k + B(-1)^k}{A4^{k-1} + B(-1)^{k-1}} = \lim_{k \rightarrow \infty} \frac{4^k}{4^{k-1}} \frac{A + B(-\frac{1}{4})^k}{A + B(-\frac{1}{4})^{k-1}}$$

$$= \boxed{4}$$

6) (10 points) Use inclusion/exclusion to prove the formula for the number $T(n, k)$ of ways to place n distinguishable balls in k distinguishable cells, with no cell empty.

$S =$ placements of the n balls into the k cells with cells possibly empty

$N = k^n$ $a_i =$ i -th cell is empty

$$\begin{aligned}
 T(n, k) &= N(a_1, a_2, \dots, a_k) = N - \sum_{i=1}^k N(a_i) + \sum_{1 \leq i < j \leq k} N(a_i a_j) \\
 &+ \dots + (-1)^p \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq k} N(a_{i_1} a_{i_2} \dots a_{i_p}) + \dots + (-1)^k N(a_1 a_2 \dots a_k) \\
 &= k^n - k(k-1)^n + \binom{k}{2}(k-2)^n + \dots + (-1)^p \binom{k}{p}(k-p)^n + \dots + (-1)^k \binom{k}{k} \cdot 0^n \\
 &= \sum_{p=0}^k (-1)^p \binom{k}{p} (k-p)^n
 \end{aligned}$$

7) (10 points) How many digraphs have exactly n vertices and exactly k oriented edges?

Possible oriented edges : $n(n-1)$

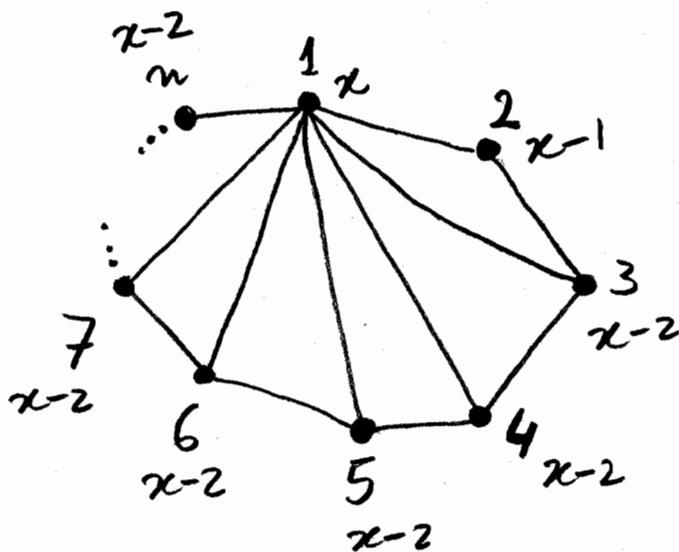
Answer: $\binom{n(n-1)}{k}$

8) (10 points) Let G be the graph obtained by modifying Z_n by adding edges connecting the vertex 1 to each one of the other vertices. In other words, G is the graph $G = (V, E)$ with

$$V = \{1, 2, \dots, n\},$$

$$E = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\} \cup \{\{1, 3\}, \{1, 4\}, \dots, \{1, n-2\}, \{1, n-1\}\}.$$

What is the chromatic polynomial of G ? (Explain your answer.)



x	choices	for vertex 1				
$x-1$	u	u	u	u	2	(distinct from vertex 1)
$x-2$	u	u	u	u	3	(u u vertices 1, 2)
$x-2$	u	u	u	u	4	(u u u 1, 3)
$x-2$	u	u	u	u	5	(u u u 1, 4)
\vdots						
$x-2$	u	u	u	u	n	(u u u 1, $n-1$)

Answer: $x(x-1)(x-2)^{n-2} = P(G, x)$

9) (10 points) Consider the network corresponding to the graph $G = (V, E)$ with weight $w(e) \geq 0$ associated to each edge $e \in E$. Define what a minimal spanning tree is (assume that we already know what a tree is).

A minimal spanning tree is a subgraph (W, F) of (V, E) with $W = V$, which is a tree and such that

$$\sum_{e \in F} w(e) \text{ is minimal among}$$

all possible choices.