MATH 113 (Winter 2005, Lecture 1)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	8	9	Total
Score										

1) (10 points) In how many ways can one place 5 indistinguishable balls into 4 distinguishable cells? (You should provide a numerical answer.)

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Answer : $\binom{8}{3} = \dfrac{8!}{5!\,3!} = \dfrac{8\times7\times6}{3\times2} = \boxed{56}$

2) (10 points) How many subsets of the set $\{1, 2, 3, 4, 5, 6, 7\}$ have an odd number of elements? (Explain how you compute your answer.)

Our set has 7 elements.

It has 2^7 subsets, half of which have an odd number of elements (because the number of subsets with an odd number of elements equals that of subsets with an even number of elements)

Answer: $\dfrac{2^7}{2} = 2^6 = \boxed{64}$

3) (10 points) Use the notation $T(n, k)$ for the number of ways to place n distinguishable balls in k distinguishable cells, with no cell empty. It is known that

$$T(n+1, k) = kT(n, k) + kT(n, k-1).$$

Use this relationship to compute $T(5, 2)$, $T(5, 3)$ and $T(5, 4)$.

$n = 1$

$n = 2$

$n = 3$

$n = 4$

$n = 5$

Answer : $T(5,2) = 30$

$T(5,3) = 150$

$T(5,4) = 240$

4) (10 points) How many codewords of length k can be written using the letters A, B, C, with A only allowed an even number of times and B only allowed an odd number of times?

Use exponential generating function (order relevant)

$$H(x) = \underbrace{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)}_{A} \underbrace{\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}_{B}$$

$$\cdot \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}_{C} = \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)e^x$$

$$= \frac{1}{4}\left(e^{2x} - e^{-2x}\right)e^x = \frac{e^{3x} - e^{-x}}{4}$$

$$= \sum_{k=0}^{\infty} \frac{1}{4}\left\{\frac{(3x)^k}{k!} - \frac{(-x)^k}{k!}\right\} = \sum_{k=0}^{\infty} \frac{3^k - (-1)^k}{4} \frac{x^k}{k!}$$

Answer: $\dfrac{3^k - (-1)^k}{4}$

5) (10 points) 5) (10 points) Suppose that the sequence (a_k) satisfies

$$a_k = 3a_{k-1} + 4a_{k-2}, \qquad a_0 = 1, \ a_1 = 2.$$

Compute

$$\lim_{k \to \infty} \frac{\ln a_k}{k}.$$

Solve recursion: $a_k = x^k$ Ch. eq. $x^2 - 3x - 4 = 0$

$$\Rightarrow \quad x = \frac{3 \pm \sqrt{9 + 16}}{2} = \frac{3 \pm 5}{2} = \begin{cases} 4 \\ -1 \end{cases}$$

General solution: $a_k = A \, 4^k + B(-1)^k$, $A, B \in \mathbb{R}$

Can compute A, B from $a_0 = 1, \ a_1 = 2$, but no need.

$$\lim_{k \to \infty} \frac{\ln a_k}{k} = \lim_{k \to \infty} \frac{\ln \left(A 4^k + B(-1)^k \right)}{k} = \lim_{k \to \infty} \frac{\ln 4^k \left(A + B(-\frac{1}{4})^k \right)}{k}$$

$$= \lim_{k \to \infty} \left\{ \ln 4 + \frac{\ln \left(A + B(-\frac{1}{4})^k \right)}{k} \right\} = \boxed{\ln 4}$$

6) (10 points) Use inclusion/exclusion to prove the formula for the number D_n of derrangement of n objects.

$S =$ set of permutations of n objects $\qquad N = n!$

$a_i :$ Object i in position i.

$$D_n = N(a_1' \, a_2' \, a_3' \dots a_n') = N - \sum_{i=1}^{n} N(a_i) + \sum_{1 \le i < j \le n} N(a_i a_j)$$

$$- \sum_{1 \le i < j < k \le n} N(a_i a_j a_k) + \dots + (-1)^k \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} N(a_{i_1} a_{i_2} \dots a_{i_k})$$

$$+ \dots + (-1)^n N(a_1 a_2 \dots a_n)$$

$$N(a_{i_1} a_{i_2} \dots a_{i_k}) = (n-k)! \quad \text{since objects } i_1, i_2, \dots i_k \text{ stay put.}$$

So $\quad D_n = N - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \dots$

$$+ (-1)^k \binom{n}{k}(n-k)! + \dots + (-1)^n (0)!$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \dots + (-1)^k \frac{n!}{k!} + \dots + (-1)^n \frac{n!}{n!}$$

$$= n! \left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^k \frac{1}{k!} + \dots + (-1)^n \frac{1}{n!} \right]$$

7) (10 points) How many digraphs have exactly n vertices and exactly k oriented edges?

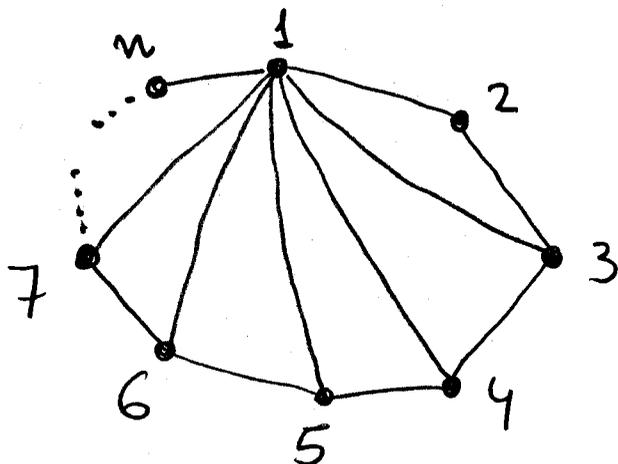Possible oriented edges : $n(n-1)$

Answer: $\dbinom{n(n-1)}{k}$

8) (10 points) Let G be the graph obtaining by modifying Z_n by adding edges connecting the vertex 1 to each one of the other vertices. In other words, G is the graph $G = (V, E)$ with

$$V = \{1, 2, ..., n\},$$

$$E = \{\{1,2\}, \{2,3\}, ...\{n-1,n\}, \{n,1\}\} \cup \{\{1,3\}, \{1,4\}, ..., \{1,n-2\}, \{1,n-1\}\}.$$

What is the chromatic number of G? (Explain your answer.)

Answer : $\chi(G) = 3$

Explanation : Color vertex 1 with one color.
Need other colors for other vertices
2 colors not enough (e.g. vertices
2 and 3 need distinct colors).
But with 3 colors can alternate
between even and odd vertices.

$\begin{cases} \text{Vertex 1: Blue} \\ \text{Odd vertices} \neq 1: \text{Red} \\ \text{Even vertices : Green} \end{cases}$

9) (10 points) Consider the network corresponding to the graph $G = (V, E)$ with weight $w(e) \geq 0$ associated to each edge $e \in E$. Define what a minimal spanning tree is (assume that we already now what a tree is).

A minimal spanning tree is a subgraph (W, F) of (V, E) with $W = V$, which is a tree and such that

$$\sum_{e \in F} w(e)$$ is minimal among

all possible choices.