MATH 113 (Spring 2004, Lecture 1)
Instructor: Roberto Schonmann
Midterm Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	Total
Score						

1) (10 points) A committee is to be chosen from among 8 seniors, 5 juniors, 2 sophomores and 3 freshmen. If the committee is to have 2 members each from a different such group, how many choices are there? (You should provide a numerical answer.)

$$\underset{8\times 5}{\text{Se J}} + \underset{8\times 2}{\text{Se So}} + \underset{8\times 3}{\text{Se F}} + \underset{5\times 2}{\text{J So}} + \underset{5\times 3}{\text{J F}} + \underset{2\times 3}{\text{So F}}$$

$$= 40 + 16 + 24 + 10 + 15 + 6$$

$$= \boxed{111}$$

2) (10 points) A fair coin is tossed 6 times. Compute the probability of getting an even number of heads and a tail on the first toss. (You should provide a numerical answer.)

$$P(E) = \frac{\#(E)}{\#(S)} = \frac{\binom{5}{0} + \binom{5}{2} + \binom{5}{4}}{2^6}$$

$$= \frac{1 + \frac{5 \times 4}{2} + 5}{64} = \frac{16}{64} = \frac{1}{4} = \boxed{0.25}$$

3) (10 points) Recall that $S(n, k)$ is the number of ways to place n distinguishable balls in k indistinguishable cells, with no cell empty. Use the notation $T(n, k)$ for the number of ways to place n distinguishable balls in k distinguishable cells, with no cell empty. What is the mathematical relationship between $S(n, k)$ and $T(n, k)$, and why does it hold?

$$T(n,k) = k!\, S(n,k)$$

Explanation: Once the balls are split into k indistinguishable cells, we can attach labels to the k cells to distinguish them. The label 1 can go to k cells, the label 2 to any of the remaining $k-1$ cells, ... so we can do the labiling of cells in $k!$ ways.

4) (10 points) How many codewords of length k can be written using the letters A, B, C, D and E, with E only allowed an odd number of times?

Let the answer be a_k and $H(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$.

$$H(x) = \underbrace{\left(1 + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}_{E}\underbrace{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^4}_{ABCD}$$

$$= \left(\frac{e^x - e^{-x}}{2}\right)(e^x)^4 = \frac{e^{5x} - e^{3x}}{2}$$

$$= \sum_{k=0}^{\infty} \frac{5^k - 3^k}{2} x^k/k!$$

$$\Rightarrow \quad a_k = \frac{5^k - 3^k}{2}.$$

5) (10 points) Three people each roll a die once. In how many ways can the score add up to 12? (You should provide a numerical answer.)

$a_k =$ # ways to get score k

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^3$$

$$= \left(\frac{x - x^7}{1-x}\right)^3 = \underbrace{x^3(1-x^6)^3}_{B(x) = \sum b_k x^k} \underbrace{\left(\frac{1}{1-x}\right)^3}_{C(x) = \sum c_k x^k}$$

$$B(x) = x^3(1 - 3x^6 + 3x^{12} - x^{18}) = x^3 - 3x^9 + 3x^{15} - x^{21}$$

$$(b_3 = 1,\ b_9 = -3,\ b_{15} = 3,\ b_{21} = -1)$$

$$C(x) = (1-x)^{-3} \Rightarrow c_k = \binom{k+3-1}{k} = \binom{k+2}{k} = \frac{(k+2)(k+1)}{2}$$

$$(a_k) = (b_k) * (c_k) \Rightarrow a_{12} = b_3 c_9 + b_9 c_3$$

$$= 1 \times \frac{11 \times 10}{2} - 3 \times \frac{5 \times 4}{2} = 55 - 30 = \boxed{25}$$