NIELSEN NUMBERS OF n-VALUED FIBER MAPS

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Abstract

The Nielsen number for n-valued multimaps, defined by Schirmer, has been calculated only for the circle. A concept of n-valued fiber map on the total space of a fibration is introduced. A formula for the Nielsen numbers of n-valued fiber maps of fibrations over the circle reduces the calculation to the computation of Nielsen numbers of single-valued maps. If the fibration is orientable, the product formula for single-valued fiber maps fails to generalize, but a "semi-product formula" is obtained. In this way, the class of n-valued multimaps for which the Nielsen number can be computed is substantially enlarged.
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1 Introduction

A *multifunction* $\phi\colon X \multimap Y$ is a function such that $\phi(x)$ is a subset of Y for each $x \in X$. For S a subset of Y, the set $\phi^{-1}(S)$ consists of the points $x \in X$ such that $\phi(x) \subseteq S$ and the set $\phi_+^{-1}(S)$ consists of the points $x \in X$ such that $\phi(x) \cap S \neq \emptyset$. A multifunction ϕ is said to be *upper semicontinuous* if U open in Y implies $\phi^{-1}(U)$ is open in X. It is *lower semicontinuous* if U open in Y implies $\phi_+^{-1}(U)$ is open in X.

An *n-valued multimap* $\phi\colon X \multimap Y$ is a function such that $\phi(x)$, for each $x \in X$, is an unordered subset of n points of Y and ϕ is both upper and lower semi-continuous. Schirmer introduced the Nielsen fixed point theory of n-valued multimaps in a series of papers [16], [17], [18]. The main result, there called the "minimum theorem"([18], Theorem 5.2) states that if $\phi\colon X \multimap X$ is an n-valued multimap of a compact triangulated manifold of dimension at least 3, then ϕ is n-valued multimap homotopic to an n-valued multimap

$\psi\colon X \multimap X$ that has exactly $N(\phi)$ fixed points, where $N(\phi)$ is the Nielsen number defined in [17].

Schirmer's papers were not concerned with the calculation of the Nielsen number. There are only two examples in those papers, both are 2-valued multimaps of the circle, for which the Nielsen number is given. In [4], in addition to extending the minimum theorem to n-valued multimaps of the circle, we determined the Nielsen numbers for all the n-valued multimaps of the circle as follows. We defined the degree of an n-valued multimap $\phi\colon S^1 \multimap S^1$ and proved that if ϕ is of degree d, then $N(\phi) = |n - d|$. This concept of degree, which is presented in Section 4 below, extends the classical definition for a single-valued map $f\colon S^1 \to S^1$ and thus the result generalizes the well-known formula $N(f) = |1 - d|$ for f of degree d.

The purpose of this paper is to expand substantially the class of n-valued multimaps for which the Nielsen number can be computed. The type of multimap we consider, which we call an n-valued fiber map, is $\Phi\colon E \multimap E$, defined on the total space of a fibration $p\colon E \to B$. In Section 2, we define n-valued fiber maps and establish their basic properties. In Section 3, we show that the covering homotopy property of fibrations holds in the setting of n-valued multimaps. As a consequence, we extend a fix-finiteness theorem for n-valued multimaps, due to Schirmer, to n-valued fiber maps. The main result of the paper, presented in Section 4, concerns fibrations in which the base space B is the circle. Under this hypothesis, we obtain an addition formula that reduces the calculation of the Nielsen number $N(\Phi)$ to the calculation of Nielsen numbers of single-valued functions. (The class of single-valued maps for which the Nielsen number can be calculated is quite large; see for instance [9] and [11].) In Section 5, we consider orientable fibrations over the circle. Nielsen numbers of single-valued fiber maps of such fibrations satisfy a product formula that does not hold in general for n-valued multimaps. However, we are able to show that a "semi-product formula"does hold and we find additional hypotheses under which the product formula is valid.

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2 Fiber maps

Throughout the paper, we will assume that all the spaces are finite polyhedra. Projections from cartesian products will be denoted $\pi_X\colon X \times Y \to X$. A map $p\colon E \to B$ is a *fibration* if it satisfies the absolute covering homotopy property. That is, given a homotopy $H\colon X \times I \to B$ and a map $f\colon X \times \{0\} \to E$ such that $pf(x) = H(x, 0)$ for all $x \in X$, there exists a homotopy $F\colon X \times I \to E$ such that $F(x, 0) = f(x)$ and $pF(x, t) = H(x, t)$ for all $(x, t) \in X \times I$.

Let $p\colon E \to B$ and $p'\colon E' \to B'$ be fibrations. A pair of maps $f\colon E \to E', \bar{f}\colon B \to B'$ such that $p'f = \bar{f}p$ is a *morphism of fibrations* ([6], p. 390), more commonly called a *fiber map* f with *induced map* $\bar{f}$ ([14], p. 75). If $\bar{f}(b) = b$ then f takes the fiber $p^{-1}(b)$ to itself and the restriction of f to the fiber is denoted by $f_b\colon p^{-1}(b) \to p^{-1}(b)$. We extend the class of fiber maps to the setting of n-valued multimaps in the following manner. An n-valued multimap $\Phi\colon E \multimap E'$ will be called an *n-valued fiber map* if there is an *induced multimap*, an n-valued multimap $\phi\colon B \multimap B'$ such that $p'\Phi = \phi p$, that is, the unordered sets $p'\Phi(e)$ and $\phi p(e)$ are identical for each $e \in E$. An n-valued fiber map Φ has the properties: (1) if $e_1, e_2 \in E$ such that $p(e_1) = p(e_2)$, then $p\Phi(e_1) = p\Phi(e_2)$ and (2) for each $e \in E$, if $\Phi(e) = \{e'_1, e'_2, \ldots, e'_n\}$, then $p(e'_i) \neq p(e'_j)$ for all $i \neq j$. If $p\colon E \to B$ is an open map, for instance a fiber bundle, then these properties are sufficient for an n-valued multimap $\Phi\colon E \multimap E'$ to be an n-valued fiber map.

An n-valued multimap $\psi\colon X \multimap Y$ is *w-split*, for some w with $2 \leq w \leq n$, if there exist n_j-valued multimaps $\psi_j\colon X \multimap Y$ for $j = 1, 2, \ldots, w$, where $n_1 + n_2 + \cdots + n_w = n$, such that

$$\psi(x) = \{\psi_1(x), \psi_2(x), \ldots, \psi_n(x)\}$$

for all $x \in X$. If ψ is an n-split n-valued multimap so that the ψ_j are single-valued maps, then ψ is just called a *split* n-valued multimap.

The following classical result from [2] is an important tool in the study of n-valued multimaps.

Lemma 2.1. *(Splitting Lemma) Let $\phi\colon X \multimap Y$ be an n-valued multimap and let*

$$\Gamma_\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}$$

be the graph of ϕ. Then $\pi_X\colon \Gamma_\phi \to X$ is a covering space. It follows that if X is simply connected, then any n-valued multimap $\phi\colon X \multimap Y$ is split.

The Splitting Lemma permits us to relate the splitting of an n-valued multimap to the structure of its graph, as follows.

Proposition 2.1. *If the graph Γ_ϕ of an n-valued multimap $\phi\colon X \multimap Y$ has w path components, then ϕ is w-split.*

Proof. Let C be a path component of Γ_ϕ then, since $\pi_X : C \to X$ is a local homeomorphism, it is a covering space by [15], Exercise 2.4, p. 151, with fibers of cardinality $n_C \leq n$. There is a multivalued function $\phi_C : X \multimap Y$ defined by $\phi_C(x) = \{y \in \phi(x) : (x, y) \in C\}$. To show that ϕ_C is lower semi-coninuous, suppose $(x_0, y_0) \in C$ and U is an open subset of Y containing y_0. Since the projection $\pi_Y : \Gamma_\phi \to Y$ is continuous, there is a neighborhood V of (x_0, y_0) in Γ_ϕ such that $(x, y) \in V$ implies $y \in U$. Let V_0 be a neighborhood of (x_0, y_0) in the open subset $V \cap C$ of Γ_ϕ such that the restriction of the covering space $\pi_X : \Gamma_\phi \to X$ to V_0 is a homeomorphism. If x is in the open subset $\pi_X(V_0)$ of X, then there exists $y \in U$ such that $(x, y) \in C$ and we have proved that ϕ_C is lower semi-continuous. For the proof that ϕ_C is upper semi-continuous, we assume that $\phi_C(x_0) \subset U$ and, for each $(x_0, y_k) \in C$, we obtain a neighborhood V_k of x_0 as we did V_0 above. Then for V the intersection of the V_k we have $\phi_C(V) \subseteq U$ to complete the proof that ϕ_C is a multimap. Now suppose that Γ_ϕ has w path components $\{C_1, C_2, \ldots, C_w\}$. Then there are finite-valued multimaps $\phi_j : X \multimap Y$ defined by $\phi_j(x) = \{y \in \phi(x) : (x, y) \in C_j\}$ such that $\phi = \{\phi_1, \phi_2, \ldots, \phi_w\}$, that is, ϕ is w-split. □

Splitting takes place for n-valued fiber maps, in the following manner.

Lemma 2.2. *Let $p : E \to B$ and $p' : E' \to B'$ be fibrations, let $\Phi : E \multimap E'$ be an n-valued fiber map and let $b \in B$, then the restriction $\Phi_b : p^{-1}(b) \multimap E'$ of Φ to $p^{-1}(b)$ is split.*

Proof. For $\phi : B \multimap B'$ the induced multimap of Φ, let $\phi(b) = \{b'_1, b'_2, \ldots, b'_n\}$. Define $f_{bj} : p^{-1}(b) \to p'^{-1}(b'_j)$ by $f_{bj}(e) = \Phi(e) \cap p'^{-1}(b'_j)$, then

$$\Phi(e) = \{f_{b1}(e), f_{b2}(e), \ldots, f_{bn}(e)\}$$

and we must show that the f_{bj} are continuous. Let $e_0 \in p^{-1}(b)$ and let u' be a neighborhood of $f_{bj}(e_0)$ in $p'^{-1}(b'_j)$. Let U' be an open subset of E' such that $U' \cap p'^{-1}(b'_j) = u'$. Let $W' \subseteq B'$ be an open subset such that $W' \cap \phi(b) = b'_j$. Since Φ is lower semi-continuous, there is an open subset V of E containing e_0 such that $\Phi(e) \cap (U' \cap p'^{-1}(W')) \neq \emptyset$ for all $e \in V$. If $e \in V \cap p^{-1}(b) = v$ then $W' \cap \phi(b) = b'_j$ implies that no point of $\Phi(e)$ other than $f_{bj}(e)$ is in $U' \cap p'^{-1}(W')$ and since $f_{bj}(e) \in p'^{-1}(b'_j)$, then it must be that $f_{bj}(e) \in u'$. Thus $f_{bj}(v) \subseteq u'$ and we conclude that f_{bj} is continuous. □

If $\Phi : E \multimap E$ is an n-valued multimap and b is a fixed point of the induced multimap $\phi : B \multimap B$, then $f_{bj} : p^{-1}(b) \to p^{-1}(b)$ for one of the $j = 1, 2, \ldots, n$. We set $f_b = f_{bj}$ for that j and we have shown that f_b is continuous.

An *n-valued homotopy* is an n-valued multimap $\Delta\colon X \times I \multimap Y$. The n-valued multimaps $\phi, \psi\colon X \multimap Y$ defined by $\phi(x) = \Delta(x, 0)$ and $\psi(x) = \Delta(x, 1)$ are said to be *homotopic*.

The path space of maps from I to a polyhedron X, with the uniform metric topology, will be denoted by X^I. Given a fibration $p\colon E \to B$, define

$$\Lambda(p) = \{(e, \omega) \in E \times B^I\colon p(e) = \omega(0)\}.$$

There is a *lifting function* for p, that is, a map $\lambda\colon \Lambda(p) \to E^I$ such that $p\lambda(e, \omega)(t) = \omega(t)$ or all $t \in I$. If p is a covering space, then it has the *unique path lifting property*, that is, if $\bar{\omega}, \bar{\omega}' \in E^I$ such that $p\bar{\omega}(t) = p\bar{\omega}'(t)$ for all $t \in I$ and $\bar{\omega}(t_0) = \bar{\omega}'(t_0)$ for some $t_0 \in I$, then $\bar{\omega}(t) = \bar{\omega}'(t)$ for all $t \in I$ ([10], Prop. 1.34, p. 62).

Generalizing Theorem 2.1 of [4], we have

Proposition 2.2. *Let $\Delta\colon X \times I \multimap Y$ be an n-valued multimap and define $\phi, \psi\colon X \multimap Y$ by $\phi(x) = \Delta(x, 0)$ and $\psi(x) = \Delta(x, 1)$. If ϕ is w-split as $\phi = \{\phi_1, \phi_2, \ldots, \phi_w\}$ then Δ is w-split as $\Delta = \{\Delta_1, \Delta_2, \ldots, \Delta_w\}$ where $\Delta_j(x, 0) = \phi_j(x)$. Therefore ψ is w-split as $\psi = \{\psi_1, \psi_2, \ldots, \psi_w\}$, where $\psi_j(x) = \Delta_j(x, 1)$, and each ϕ_j is homotopic to ψ_j by Δ_j.*

Proof. We consider the covering space $\pi_{X\times I}\colon \Gamma_\Delta \to X \times I$ and its lifting function $\lambda\colon \Lambda(\pi_{X\times I}) \to (\Gamma_\Delta)^I$. For $1 \le j \le w$, define $\Delta_j\colon X \times I \multimap Y$ as follows: if $\phi_j(x) = \{y_1, y_2, \ldots, y_{n_j}\}$, then

$$\Delta_j(x, t) = \{\pi_Y[\lambda(((x, 0), y_i), x \times I)(t)]\}_{i=1}^{n_j}.$$

The unique path lifting property implies that $\Delta_j\colon X \times I \multimap Y$ is n_j-valued and that if $j \ne k$, then $\Delta_j(x, t) \cap \Delta_k(x, t) = \emptyset$. Since $n_1 + n_2 + \cdots + n_w = n$, we see that $\{\Delta_1, \Delta_2, \ldots, \Delta_w\}$ is the required w-splitting of Δ. □

Proposition 2.3. *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be fibrations and let $\Phi\colon E \multimap E'$ be an n-valued fiber map with induced multimap $\phi\colon B \multimap B'$. If ϕ is w-split as $\phi = \{\phi_1, \phi_2, \ldots, \phi_w\}$, then Φ is w-split as $\Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_w\}$ such that each ϕ_j is the induced multimap of the n_j-valued fiber map Φ_j.*

Proof. For $j = 1, 2, \ldots, w$ define $\Phi_j\colon E \multimap E'$ by

$$\Phi_j(e) = \Phi(e) \cap p'^{-1}(\phi_j(p(e)))$$

then $\Phi(e) = \{\Phi_1(e), \Phi_2(e), \ldots, \Phi_w(e)\}$ and $p'\Phi_j = \phi_j p$. We will prove that the Φ_j are continuous. Let $e_0 \in E$ and set $b_0 = p(e_0)$. There are disjoint open subsets $W'_1, W'_2, \ldots, W'_w$ of B' such that $\phi_j(b_0) \subseteq W'_j$ for $j = 1, 2, \ldots, w$. Since the ϕ_j are upper semi-continuous, there are neighborhoods $V_1, V_2, \ldots, V_w$ of b_0 such that $\phi_j(V_j) \subseteq W'_j$. Let $V = V_1 \cap V_2 \cap \cdots \cap V_w$ then for $b \in V$ we have $\phi_j(b) \subseteq W'_j$ and $\phi_k(b) \cap W'_j = \emptyset$ if $k \neq j$. Now let U' be an open subset of E' such that $\Phi_j(e_0) \cap U' \neq \emptyset$. Let $U'_j = U' \cap p'^{-1}(W'_j)$. Since Φ is lower semi-continuous, there is a neighborhood $\mathcal{O}$ of e_0 such that $e \in \mathcal{O}$ implies $\Phi(e) \cap U'_j \neq \emptyset$. If $e \in \mathcal{O} \cap p^{-1}(V)$, then it must be that $\Phi_k(e) \cap U'_j = \emptyset$ for $k \neq j$ because $\phi_k(p(e)) \cap W'_j = \emptyset$. We have proved that $\Phi_j(\mathcal{O} \cap p^{-1}(V)) \cap U'_j \neq \emptyset$ and therefore $\Phi_j(\mathcal{O} \cap p^{-1}(V)) \cap U' \neq \emptyset$ so Φ_j is lower semi-continuous. The upper semi-continuity of Φ_j is a consequence of the corresponding property of Φ in the same manner. □

3 Fix-finiteness

The main result, Theorem 6, of [16], a generalization of a classical result for single-valued maps due to Hopf [13], states that an n-valued multimap $\phi\colon X \multimap X$ on a finite polyhedron can be approximated arbitrarily closely by an n-valued multimap with only finitely many fixed points, each of them in a maximal simplex of the polyhedron. According to Lemma 4.1 of [17], sufficiently close n-valued multimaps are homotopic. Thus $\phi\colon X \multimap X$ is homotopic to an n-valued multimap $\psi\colon X \multimap X$ that is *fix-finite* that is, it has finitely many fixed points, and each fixed point lies in a maximal simplex of X. The purpose of the present section is to prove the corresponding result in the setting of n-valued fiber maps.

We first extend the covering homotopy property to n-valued multimaps in order to obtain a tool that we will need for the study of n-valued fiber maps.

Theorem 3.1. *Let X be a finite polyhedron, $p\colon E \to B$ a fibration, $\phi\colon X \multimap E$ an n-valued multimap and $\delta\colon X \times I \multimap B$ an n-valued multimap such that $p\phi(x) = \delta(x, 0)$ for all $x \in X$, then there exists an n-valued multimap $\Delta\colon X \times I \multimap E$ such that $\Delta(x, 0) = \phi(x)$ and $p\Delta(x, t) = \delta(x, t)$ for all $(x, t) \in X \times I$.*

Proof. Choose $b_0 \in B$ and set $Y = p^{-1}(b_0)$. Define an open cover of $X \times I$ as follows. For $(x, t) \in X \times I$, write $\delta(x, t) = \{b_1, b_2, \ldots, b_n\}$ and let $U_1, U_2, \ldots, U_n$ be n disjoint open contractible subsets of B, such that $U_j \cap \delta(x, t) = b_j$ for each $j = 1, 2, \ldots, n$. By Corollary

2.8.15 of [19], there are homotopy equivalences $\zeta_j\colon p^{-1}(U_j) \to U_j \times Y$ with $\pi_{U_j}\zeta_j = p$ and $\theta_j\colon U_j \times Y \to p^{-1}(U_j)$ with $p\theta_j = \pi_{U_j}$. Since δ is upper semi-continuous with respect to the product topology on $X \times I$, there are open subsets V of X and J of I such that $(x,t) \in V \times J$ and if $(x',t') \in V \times J$ then $\delta(x',t') \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$. Moreover, since X is locally contractible, we will choose $V \times J$ to be contractible. By the Splitting Lemma, the restriction of δ to $V \times J$ splits into n maps $f_j\colon V \times J \to B$ where we number the f_j so that $f_j(V \times J) \subseteq U_j$. Since $X \times I$ is compact, there is a finite subcover $\{V_k \times J_\mu\}$ of the cover $\{V \times J\}$. By writing open intervals in I as unions of smaller intervals open in I, if necessary, we can choose the J_μ so that each intersects at most two others and number them $\{J_1, J_2, \ldots, J_r\}$ so that J_μ intersects only $J_{\mu-1}$ and $J_{\mu+1}$. Choose $0 = t_1 < t_2 < \cdots < t_r < t_{r+1} = 1$ such that $t_\mu \in J_{\mu-1} \cap J_\mu$ for $\mu = 2, \ldots, r$.

Now $\Delta(x, t_1) = \phi(x)$ is defined by hypothesis, so we assume that $\Delta\colon X \times [0, t_\mu] \multimap E$ has been defined and we will extend Δ over $X \times [t_\mu, t_{\mu+1}]$. Subdivide X so that the mesh of the triangulation is less than the Lebesgue number of the cover $\{V_k\}$. Let x be a vertex, then the restriction of δ to $x \times [t_\mu, t_{\mu+1}]$ splits as maps $d_j\colon x \times [t_\mu, t_{\mu+1}] \to B$ for $j = 1, 2, \ldots, n$. Since $x \times [t_\mu, t_{\mu+1}]$ is contained in $V_k \times J_\mu$ for some V_k, each $d_j(x \times [t_\mu, t_{\mu+1}]) \subset U_j$ for disjoint contractible neighborhoods U_j. Define $\Delta\colon x \times [t_\mu, t_{\mu+1}] \multimap E$ by

$$\Delta(x,t) = \{\theta_j[d_j(x,t), \pi_Y\zeta_j(\Delta(x,t_\mu) \cap p^{-1}(U_j))]\}_{j=1}^n.$$

Note that $\Delta(x,t)$ is well-defined because $\delta(x, t_\mu) \cap U_j$ is a single point for each j. In this way, Δ is extended to $[t_\mu, t_{\mu+1}]$ over the zero-skeleton of X. Now assume that Δ has been extended over the $(m-1)$-skeleton of X and let σ be an m-simplex. There is a retraction $\rho\colon \sigma \times [t_\mu, t_{\mu+1}] \to T$ where

$$T = (\sigma \times t_\mu) \cup (\partial\sigma \times [t_\mu, t_{\mu+1}]).$$

The restriction of δ to T splits as maps $\{d_j\colon T \to U_j\}$ to disjoint contractible neighborhoods U_j. We extend Δ to $\sigma \times [t_\mu, t_{\mu+1}]$ by setting

$$\Delta(x,t) = \{\theta_j[d_j(x,t), \pi_Y\zeta_j(\Delta(\rho(x,t)) \cap p^{-1}(U_j))]\}_{j=1}^n.$$

The required n-valued multimap $\Delta\colon X \times I \multimap E$ is thus defined by induction. □

An n-valued fiber map $\Delta\colon E \times I \multimap E$ is an *n-valued fiber homotopy.* We say that the n-valued fiber maps $\Phi, \Psi\colon E \multimap E$ defined by $\Phi(e) = \Delta(e, 0)$, $\Psi(e) = \Delta(e, 1)$ are *fiber homotopic.*

Theorem 3.2. *Let $p\colon E \to B$ be a fibration and let $\Phi\colon E \multimap E$ be an n-valued fiber map, then there is a fix-finite n-valued fiber map $\Psi\colon E \multimap E$ fiber homotopic to Φ.*

Proof. As noted above, Theorem 6 of [16] and Lemma 4.1 of [17] imply that there is an n-valued homotopy $\delta\colon B \times I \multimap B$ such that $\delta(b,0) = \phi(b)$, where ϕ is the induced multimap of Φ, and $\delta(b,1) = \psi(b)$ defines a fix-finite n-valued multimap. By Theorem 6 of [16], the fixed points of ψ lie in maximal simplices of B. By Theorem 3.1, there is an n-valued multimap $\Delta\colon E \times I \multimap E$ such that $\Delta(e,0) = \Phi(e)$ and $p\Delta(e,t) = \delta(p(e),t)$ for all $(e,t) \in E \times I$. Then $\Gamma\colon E \multimap E$ defined by $\Gamma(e) = \Delta(e,1)$ is an n-valued fiber map with induced multimap ψ so it has fixed points only in finitely many fibers. Since Γ is fiber homotopic to Φ, we may assume, to simplify the notation, that the n-valued fiber map Φ of the statement of the theorem has fixed points only in finitely many fibers.

Let $p^{-1}(b)$ be one such fiber, that is, $b \in \phi(b)$ is a fixed point of ϕ. Let U be a contractible neighborhood of b containing no other fixed point of ϕ. By the Splitting Lemma, $\phi = \{\phi_1, \phi_2, \ldots, \phi_n\}\colon U \multimap B$ and so, by Proposition 2.3, $\Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}\colon p^{-1}(U) \multimap E$. We may assume $\phi_1(b) = b$, so $\Phi_1(p^{-1}(b)) \subset p^{-1}(b)$. Let $h\colon p^{-1}(b) \times I \to p^{-1}(b)$ such that $h(e,0) = \Phi_1(e)$ and $h_1(e) = h(e,1)$ is a fix-finite map. Let V be a neighborhood of b whose closure lies in U and define a subset T of $p^{-1}(U) \times I$ by

$$T = (p^{-1}(U) \times \{0\}) \cup (p^{-1}(U \setminus V) \times I) \cup (p^{-1}(b) \times I).$$

Define $H\colon T \to E$ by

$$H(e,t) = \begin{cases} \Phi_1(e) & \text{if } t = 0 \text{ or } e \in p^{-1}(U \setminus V) \\ h(e,t) & \text{if } e \in p^{-1}(b) \end{cases}$$

By the fiber homotopy extension theorem ([1], Theorem 2.2), we can extend H to a fiber-preserving homotopy $H\colon p^{-1}(U) \times I \to E$. Define $\Gamma_1\colon p^{-1}(U) \to E$ by $\Gamma_1(e) = H(e,1)$. Noting that $\Gamma_1(e) = \Phi_1(e)$ for $e \in p^{-1}(U \setminus V)$, we define $\Psi\colon E \multimap E$ by setting

$$\Psi(e) = \{\Gamma_1(e), \Phi_2(e), \ldots, \Phi_n(e)\}$$

for $e \in p^{-1}(U)$ and $\Psi(e) = \Phi(e)$ otherwise. In the same way, the homotopy may be extended to an n-valued fiber homotopy between the n-valued fiber maps Ψ and Φ. Repeating this construction for each of the finite number of fixed points of ϕ completes the proof. □

4 An addition formula

Throughout the rest of the paper, all spaces are finite polyhedra that are connected.

In [17], Schirmer used the Splitting Lemma to generalize Nielsen's definition of equivalence of fixed points of maps in the following way. For an n-valued multimap $\phi\colon X \multimap X$, let $Fix(\phi) = \{x \in X\colon x \in \phi(x)\}$. Then $x_0, x_1 \in Fix(\phi)$ are *equivalent* if there is a path $c\colon I \to X$ with $c(0) = x_0$ and $c(1) = x_1$ and a map $\phi_j\colon I \to X$ of the splitting $\phi c = \{\phi_1, \phi_2, \ldots, \phi_n\}\colon I \multimap X$ such that $\phi_j(0) = x_0, \phi_j(1) = x_1$ and ϕ_j is homotopic to c relative to the endpoints. An equivalence class is called a *fixed point class*.

The fixed point index of ϕ at an isolated fixed point x, denoted $ind(\phi, x)$ is defined in [17], page 210 in terms of the classical fixed point index by $ind(\phi, x) = ind(\phi_j, x)$ where $\phi = \{\phi_1, \phi_2, \ldots, \phi_n\}$ is a splitting of ϕ in a neighborhood of x and $\phi_j(x) = x$. Let U be a neighborhood of a fixed point class $\mathbf{F}$, then Theorem 6 of [16] approximates the restriction of ϕ to U by a fix-finite n-valued multimap and the index $ind(\mathbf{F})$ of the fixed point class $\mathbf{F}$ is defined to be the sum of the indices of the fixed points of the approximation. The Nielsen number $N(\phi)$ is the number of *essential* fixed point classes, that is, those of nonzero index.

The following result is a consequence of Theorem 4.1 of [12]. [1]

Proposition 4.1. *Let $p\colon E \to S^1$ be a fibration and let $f\colon E \to E$ be a fiber map with induced map $\bar{f}\colon S^1 \to S^1$ of degree d. If $d = 1$, then $N(f) = 0$. Otherwise, let $b_1, b_2, \ldots, b_{|1-d|}$ be points of S^1 such that each b_j is in a different essential fixed point class of $\bar{f}$, then*

$$N(f) = \sum_{j=1}^{|1-d|} N(f_{b_j}).$$

Let $p\colon E \to B$ be a fibration. Since B is a polyhedron, there is a *regular* lifting function for p, that is, a map $\lambda\colon \Lambda(p) \to E^I$ such that $p\lambda(e, \omega)(t) = \omega(t)$ for all $t \in I$ and, in addition, if ω is a constant path, so also is $\lambda(e, \omega)$. For a path $\omega \in B^I$ there is a map $\tau_\omega\colon p^{-1}(\omega(0)) \to p^{-1}(\omega(1))$, called the *fiber translation* obtained from ω, defined by $\tau_\omega(e) = \lambda(e, \omega)(1)$. If $\omega' \in B^I$ such that $\omega'(0) = \omega(0), \omega'(1) = \omega(1)$ and ω and ω' are homotopic relative to the endpoints, then the fiber translations τ_ω and $\tau_{\omega'}$ are homotopic. Therefore a fiber translation τ_ω is a homotopy equivalence with homotopy inverse $\tau_{\bar{\omega}}$ where $\bar{\omega}(t) = \omega(1 - t)$.

[1] A thorough survey of the Nielsen theory of single-valued fiber maps is presented by Heath in [11].

Lemma 4.1. *Let $p\colon E \to B$ be a fibration and let $\Phi\colon E \multimap E$ be an n-valued fiber map with induced multimap $\phi\colon B \multimap B$. If $b_0, b_1 \in Fix(\phi)$ are equivalent, then $N(f_{b_0}) = N(f_{b_1})$.*

Proof. Since b_0 and b_1 are equivalent, there is a path $c \in B^I$ with $c(0) = b_0$ and $c(1) = b_1$ and a splitting $\phi c = \{\phi_1, \phi_2, \ldots, \phi_n\}\colon I \multimap B$ with a path ϕ_j such that $\phi_j(0) = b_0, \phi_j(1) = b_1$ and ϕ_j is homotopic to c relative to the endpoints. For $t \in I$, define a path $\phi_j^{[t]}\colon I \to B$ by $\phi_j^{[t]}(s) = \phi_j(s+t)$ for $0 \le s \le 1-t$ and $\phi_j^{[t]}(s) = \phi_j(1) = b_1$ for $1-t \le s \le 1$. Define $H\colon p^{-1}(b_0) \times I \to p^{-1}(b_1)$ by

$$H(e,t) = \lambda([\Phi(\lambda(e,c)(t)) \cap p^{-1}(\phi_j(t))], \phi_j^{[t]})(1).$$

The function H is well-defined because

$$p(\Phi(\lambda(e,c)(t)) = \phi(p(\lambda(e,c)(t)) = \phi c(t)$$

and it is continuous by Lemma 2.2. Now

$$\begin{aligned} H(e,0) &= \lambda([\Phi(\lambda(e,c)(0) \cap p^{-1}(\phi_j(0))], \phi_j^{[0]})(1) \\ &= \lambda([\Phi(e) \cap p^{-1}(b_0)], \phi_j)(1) \\ &= \lambda(f_{b_0}(e), \phi_j)(1) = \tau_{\phi_j} f_{b_0}(e) \end{aligned}$$

and

$$\begin{aligned} H(e,1) &= \lambda([\Phi(\lambda(e,c)(1) \cap p^{-1}(\phi_j(1))], \phi_j^{[1]})(1) \\ &= \lambda([\Phi(\tau_c(e)) \cap p^{-1}(b_1)], \phi_j^{[1]})(1) \\ &= \lambda(f_{b_1}\tau_c(e), \phi_j^{[1]})(1) = f_{b_1}\tau_c(e) \end{aligned}$$

so $\tau_{\phi_j} f_{b_0}$ and $f_{b_1}\tau_c$ are homotopic. Since ϕ_j is homotopic to c, then $\tau_{\phi_j} f_{b_0}$ is homotopic to $\tau_c f_{b_0}$ and therefore f_{b_0} is homotopic to $\tau_{\bar{c}} f_{b_1} \tau_c$. By Theorem 5.4 of [14], this implies that $N(f_{b_0}) = N(f_{b_1})$. $\square$

Lemma 4.2. *Let $p\colon E \to B$ be a fibration and let $\Phi, \Psi\colon E \multimap E$ be fiber homotopic n-valued fiber maps with induced multimaps $\phi, \psi\colon B \multimap B$. There is a one-to-one correspondence between the essential fixed point classes of ϕ and ψ such that if $b_0 \in Fix(\phi)$ and $b_1 \in Fix(\psi)$ are in corresponding essential fixed point classes, then $N(f_{b_0}) = N(g_{b_1})$ where $f_{b_0}\colon p^{-1}(b_0) \to p^{-1}(b_0)$ and $g_{b_1}\colon p^{-1}(b_1) \to p^{-1}(b_1)$ are defined by $f_{b_0}(e) = \Phi(e) \cap p^{-1}(b_0)$ and $g_{b_1}(e) = \Psi(e) \cap p^{-1}(b_1)$.*

Proof. Let $D\colon E \times I \multimap E$ be an n-valued fiber map with induced multimap $d\colon B \times I \multimap B$ such that $D(e,0) = \Phi(e), D(e,1) = \Psi(e)$

and d is an n-valued homotopy between ϕ and ψ. Define $\mathbf{D}\colon E \times I \multimap E \times I$ as follows: if $D(e,t) = \{e_1, e_2, \ldots e_n\}$, then $\mathbf{D}(e,t) = \{(e_1,t),(e_2,t),\ldots(e_n,t)\}$. Then $\mathbf{D}$ is an n-valued fiber map of the fibration $p_\times\colon E \times I \to B \times I$ defined by $p_\times(e,t) = (p(e),t)$ with induced multimap $\mathbf{d}\colon B \times I \multimap B \times I$. Let $\mathbf{F}_0$ be an essential fixed point class of ϕ then, by Lemma 6.3 of [17], there is a unique fixed point class $\mathbf{F}$ of $\mathbf{d}$ such that $\mathbf{F}_0 = \{b \in B\colon (b,0) \in \mathbf{F}\}$. By Lemmas 6.2 and 6.4 of [17], there is an essential fixed point class $\mathbf{F}_1$, the corresponding class, such that $\mathbf{F}_1 = \{b \in B\colon (b,1) \in \mathbf{F}\}$. Let $b_0 \in \mathbf{F}_0$ and $b_1 \in \mathbf{F}_1$ then $(b_0,0)$ and $(b_1,1)$ are equivalent fixed points of $\mathbf{d}$. Applying Lemma 4.1 to $\mathbf{D}\colon E \times I \multimap E \times I$, we conclude that $N(\delta_{b_0}) = N(\delta_{b_1})$ where, for $v = 0,1$, the map δ_{b_v} is defined by $\delta_{b_v}(e,v) = \mathbf{D}(e,v) \cap p_\times^{-1}(b_v,v)$. Since $\mathbf{D}(e,0) = (\Phi(e),0)$ and $p_\times^{-1}(b_0) = p^{-1}(b_0) \times \{0\}$, then $N(\delta_{b_0}) = N(f_{b_0})$. Similarly, $N(\delta_{b_1}) = N(g_{b_1})$, so $N(f_{b_0}) = N(g_{b_1})$. □

In [4], the *degree* $deg(\phi)$ of an n-valued multimap $\phi\colon S^1 \multimap S^1$ is defined as follows. Let $\eta\colon \mathbf{R} \to S^1$ be the universal covering space defined by $\eta(t) = e^{i2\pi t}$. There is a splitting $\{f_0, f_1, \ldots, f_{n-1}\}$ of $\phi\eta\colon \mathbf{R} \multimap S^1$ where the f_j are ordered so that $f_j(0) = \eta(t_j)$ for $0 \leq t_0 < t_1 < \cdots < t_{n-1} < 1$. Let $\tilde{f}_0$ be the lift of f_0 to t_0, then $\tilde{f}_0(1) = v + t_J$ for some integer v and $0 \leq J \leq n-1$. The degree of ϕ is defined by $deg(\phi) = nv + J$.

To illustrate the definition, represent S^1 as the unit circle in the complex plane and define $\phi\colon S^1 \multimap S^1$ by $\phi(z) = \{z^{3/2}\}$. In terms of the covering space η,

$$\phi(\eta(t)) = \{\eta(\frac{3}{2}t), \eta(\frac{3}{2}t + \frac{1}{2})\}$$

so $\phi\eta = \{f_0, f_1\}$ where $f_0(t) = \eta(\frac{3}{2}t)$ and $f_1(t) = \eta(\frac{3}{2}t + \frac{1}{2})$ and thus $t_0 = 0$ and $t_1 = \frac{1}{2}$. Now $\tilde{f}_0(t) = \frac{3}{2}t$ so

$$\tilde{f}_1(1) = \frac{3}{2} = 1 + \frac{1}{2} = v + t_1$$

and therefore $deg(\phi) = nv + J = 2(1) + 1 = 3$.

An equivalent definition of $deg(\phi)$ that resembles the definition of degree for single-valued maps is developed in [5]. A homomorphism of integer homology $\phi_*\colon H_1(S^1) \to H_1(S^1)$ is induced by ϕ in such a way that $\phi_*(1) = deg(\phi) \cdot 1$, where 1 is a generator of $H_1(S^1)$. However, we will need the covering space definition of the degree to prove the following result.

Lemma 4.3. *Let $p\colon E \to S^1$ be a fibration, let $\Phi\colon E \multimap E$ be an n-valued fiber map and let $b_0 \in S^1$ be a fixed point of the induced* *multimap* $\phi\colon S^1 \multimap S^1$. *If* $deg(\phi) \neq n$, *then* $\mathbf{f}$ *is a fixed point class of* $f_{b_0}\colon p^{-1}(b_0) \to p^{-1}(b_0)$ *if and only if there is a fixed point class* $\mathbf{F}$ *of* Φ *such that* $\mathbf{F} \cap p^{-1}(b_0) = \mathbf{f}$.

Proof. Suppose e_0 and e_1 are equivalent fixed points of f_{b_0} in a fixed point class $\mathbf{f}$, then there exists a map $c\colon I \to p^{-1}(b_0)$ such that $c(0) = e_0, c(1) = e_1$ and $f_{b_0}c$ is homotopic to c relative to the endpoints. Since, by Lemma 2.2, the restriction Φ_0 of $p^{-1}(b_0)$ splits with f_{b_0} as a factor, then e_0 and e_1 are equivalent fixed points of Φ. Therefore, there is a fixed point class $\mathbf{F}$ of Φ such that $\mathbf{f} \subseteq \mathbf{F}$.

To prove that $\mathbf{F} \cap p^{-1}(b_0) \subseteq \mathbf{f}$, let $e_0, e_1 \in \mathbf{F} \cap p^{-1}(b_0)$. Therefore there is a path $c\colon I \to E$ such that $c(0) = e_0, c(1) = e_1$ and a splitting $\Phi c = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}\colon I \multimap E$ such that the paths c and Φ_1 are homotopic relative to the endpoints. We must prove that e_0 and e_1 are equivalent fixed points of f_{b_0}.

We have a splitting $\{p\Phi_1, p\Phi_2, \ldots, p\Phi_n\}\colon I \multimap S^1$ of $p\Phi c = \phi(pc)$ such that pc is homotopic to $p\Phi_1$. We first assume pc is contractible, so there is a map $h\colon I \times I \to B$ that contracts pc to the constant loop at b_0. By the covering homotopy property, we may lift h to c to obtain a homotopy $H\colon I \times I \to E$, relative to the endpoints, between c and a path $c_0\colon I \to p^{-1}(b_0)$. Let $\Phi H = \{\Phi_1^*, \Phi_2^*, \ldots, \Phi_n^*\}$ be a splitting such that the restriction of Φ_1^* is Φ_1. Then $\Phi_1^*\colon I \times I \to E$ is a homotopy between Φ_1 and $f_{b_0}c_0$. Since c is homotopic to Φ_1, we conclude that c_0 and $f_{b_0}(c_0)$ are paths in $p^{-1}(b_0)$ from e_0 to e_1 that are homotopic in E. In a fibration over S^1, fibers are π_1-injective into the total space, so c_0 and $f_{b_0}(c_0)$ are paths that are homotopic in $p^{-1}(b_0)$ and therefore e_0 and e_1 are equivalent fixed points of f_{b_0}.

Now suppose that the loop pc is not contractible. We claim that if pc is homotopic to $\phi_1 = p\Phi_1$, then the degree of ϕ must be n and thus, since $deg(\phi) \neq n$ by hypothesis, no such loop exists and therefore e_0 and e_1 must have been equivalent as fixed points of f_{b_0} by the previous step, which will complete the proof. Let $\eta\colon \mathbf{R} \to S^1$ be the covering space and let $\widetilde{pc}\colon I \to \mathbf{R}$ be the lift of pc to the origin. Since pc is a loop, then $\widetilde{pc}(1) = m$ for some integer m, which is nonzero since pc is not contractible. Lifting the homotopy between pc and ϕ_1 to the origin, the lift $\tilde{\phi}_1$ of ϕ_1 must have the property $\tilde{\phi}_1(1) = m$ also. Since m is an integer, $J = 0$ in the definition of the degree of ϕ and therefore ϕ is split by Corollary 5.1 of [4]. Thus $\phi_1\colon S^1 \to S^1$ is a well-defined map such that the homomorphism $\phi_{1\pi}\colon \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ fixes the non-zero element $[pc] \in \pi_1(S^1, b_0)$. The map ϕ_1 must therefore be of degree one so, by Theorem 2.2 of [4], $deg(\phi) = n \cdot 1 = n$. □

We may now generalize Proposition 4.1 as follows.

Theorem 4.1. *(Addition Formula) Let $p\colon E \to S^1$ be a fibration and let $\Phi\colon E \multimap E$ be an n-valued fiber map with induced multimap $\phi\colon S^1 \multimap S^1$ of degree d. If $d = n$, then $N(\Phi) = 0$. Otherwise, let $b_1, b_2, \ldots, b_{|n-d|} \in S^1$, such that each b_j is in a different essential fixed point class of ϕ, then*

$$N(\Phi) = \sum_{j=1}^{|n-d|} N(f_{b_j}).$$

Proof. If $d = n$ then, by Theorem 5.1 of [4], there is a homotopy of ϕ to a fixed point free n-valued multimap ψ. By Theorem 3.1, there is an n-valued fiber map $\Psi\colon E \multimap E$ homotopic to Φ such that ψ is the induced multimap of Ψ. Since ψ has no fixed points, Ψ is also fixed point free so $N(\Psi) = 0$, and therefore $N(\Phi) = 0$ by Theorem 6.5 of [17].

Now we assume that $d \neq n$. By Theorems 3.1 and 4.1 of [4], ϕ is homotopic to the n-valued power map $\phi_{n,d}\colon S^1 \multimap S^1$ that has fixed points $\beta_1, \beta_2, \ldots, \beta_{|n-d|}$, all of index +1 or all of index -1, such that no two of these fixed points are equivalent. By the proof of Theorem 3.2, there is a fix-finite n-valued fiber map $\Psi\colon E \multimap E$ that is fiber homotopic to Φ and the induced multimap of Φ is $\phi_{n,d}$. Let β_j be a fixed point of $\phi_{n,d}$ and let $\mathbf{f} = \{e_1, e_2, \ldots, e_r\}$ be an essential fixed point class of $g_{\beta_j}\colon p^{-1}(\beta_j) \to p^{-1}(\beta_j)$ defined by $g_{\beta_j}(e) = \Psi(e) \cap p^{-1}(\beta_j)$. By Lemma 4.3, there is a fixed point class $\mathbf{F}$ of Ψ such that $\mathbf{F} \cap p^{-1}(\beta_j) = \mathbf{f}$. If $\mathbf{F} \cap p^{-1}(\beta_k) \neq \emptyset$ for some $k \neq j$, let $e_0 \in \mathbf{F} \cap p^{-1}(\beta_k)$, then there is a path $c\colon I \to E$ such that $c(0) = e_0$ and $c(1) = e_1 \in \mathbf{f}$ such that, for a splitting $\Psi c = \{\Psi_1, \Psi_2, \ldots, \Psi_n\}\colon I \multimap E$, some path Ψ_u is homotopic to c relative to the endpoints, which are fixed by Ψ_u. But then there is a splitting $p\Psi c = \phi_{n,d}(pc) = \{p\Psi_1, p\Psi_2, \ldots, p\Psi_n\}\colon I \multimap B$ such that $p\Psi_u$ is homotopic to pc relative to the endpoints β_k and β_j, which would imply that these are equivalent fixed points of $\phi_{n,d}$. Therefore, $\mathbf{f} = \mathbf{F}$ is a fixed point class of Ψ.

Let $U \subseteq S^1$ be a closed interval with β_j in the interior and $U \cap \beta_k = \emptyset$ for $k \neq j$. The restriction of $\phi_{n,d}$ to U splits as $\{\psi_1^U, \psi_2^U, \ldots, \psi_n^U\}\colon U \multimap S^1$. By Proposition 2.3, the restriction of Ψ to $p^{-1}(U)$ splits as $\{\Psi_1^U, \Psi_2^U, \ldots, \Psi_n^U\}\colon p^{-1}(U) \multimap E$ with each ψ_j^U the induced multimap of Ψ_j^U. Let $\mathbf{f} \subseteq p^{-1}(\beta_j)$ be an essential fixed point class of g_{β_j} and let $e_v \in \mathbf{f}$, then $\Psi_k^U(e_v) = e_v$ for some k.

By Lemma 3.1 on page 84 of [14], the index of Ψ at e_v satisfies

$$\begin{aligned} ind(\Psi, e_v) &= ind(\Psi_k^U, e_v) \\ &= ind(\psi_k^U, \beta_j) \cdot ind(g_{\beta_j}, e_v) \\ &= ind(\phi_{n,d}, \beta_j) \cdot ind(g_{\beta_j}, e_v). \end{aligned}$$

Since $ind(\mathbf{f}) \neq 0$, then

$$\begin{aligned} ind(\mathbf{F}) &= \sum_{v=1}^{r} ind(\Psi, e_v) \\ &= ind(\phi_{n,d}, \beta_j) \sum_{v=1}^{r} ind(g_{\beta_j}, e_v) \\ &= \pm \sum_{v=1}^{r} ind(g_{\beta_j}, e_v) = \pm\, ind(\mathbf{f}) \neq 0 \end{aligned}$$

so $\mathbf{f} = \mathbf{F}$ is also an essential fixed point class of Ψ.

Conversely, if $\mathbf{F}$ is an essential fixed point class of Ψ such that $\mathbf{F} \cap p^{-1}(\beta_j) \neq \emptyset$, then $\mathbf{f} = \mathbf{F} \cap p^{-1}(\beta_j)$ is a fixed point class of g_{β_j} by Lemma 4.3 and, again, $ind(\mathbf{F}) = \pm\, ind(\mathbf{f})$ so $\mathbf{f}$ is also an essential fixed point class of g_{β_j}. We conclude that the essential fixed point classes of Ψ are the essential fixed point classes of the g_{β_j} for $j = 1, 2, \ldots, |n-d|$ so

$$N(\Psi) = \sum_{j=1}^{|n-d|} N(g_{\beta_j}).$$

Since Φ is fiber homotopic to Ψ, by Lemma 4.2 there is a one-to-one correspondence between the fixed points $\beta_1, \beta_2, \ldots, \beta_{|n-d|}$ of $\phi_{n,d}$ and the essential fixed point classes of ϕ such that, if we choose $b_1, b_2, \ldots, b_{|n-d|} \in S^1$ in each of the corresponding essential fixed point classes of ϕ, then $N(f_{b_j}) = N(g_{\beta_j})$ for $j = 1, 2, \ldots, |n-d|$. Since $N(\Phi) = N(\Psi)$ by Theorem 6.5 of [17], we have proved that

$$N(\Phi) = \sum_{j=1}^{|n-d|} N(f_{b_j}).$$

□

To illustrate the addition formula, let $\mathbf{R}^2$ denote the plane and represent the Klein bottle as $K = \mathbf{R}^2/\sim$ where $(x, y) \sim (x + k, (-1)^k y)$ and $(x, y) \sim (x, y + \ell)$ for all integers k, ℓ. The projection $\tilde{p}\colon \mathbf{R}^2 \to \mathbf{R}$ on the first factor induces the fibration $p\colon K \to S^1$

with fiber S^1, where $S^1 = \mathbf{R}/\sim$ with $x \sim x + k$. Define a 2-valued multimap $\widetilde{\Phi}\colon \mathbf{R}^2 \multimap \mathbf{R}^2$ by

$$\widetilde{\Phi}(x, y) = \{(-x + 1, 2y), (-x + \tfrac{3}{2}, 3y)\}$$

then $\widetilde{\Phi}$ induces a 2-valued fiber map $\Phi\colon K \multimap K$. The induced multimap $\phi\colon S^1 \multimap S^1$ has four fixed points $v = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$; let $f_v\colon p^{-1}(v) \to p^{-1}(v)$ be the corresponding restrictions of Φ. Since $\widetilde{\Phi}(0, y) = \{(1, 2y), (\frac{3}{2}, 3y)\}$ and $(1, 2y) \sim (0, -2y)$, the degree of f_0 is -2. Similarly, $f_{\frac{1}{4}}$ is of degree -3. Furthermore, $\widetilde{\Phi}(\frac{1}{2}, y) = \{(\frac{1}{2}, 2y), (1, 3y)\}$ so $f_{\frac{1}{2}}$ is of degree 2 and, similarly, $f_{\frac{3}{4}}$ is of degree 3. Therefore, by Theorem 2.4,

$$N(\Phi) = N(f_0) + N(f_{\frac{1}{4}}) + N(f_{\frac{1}{2}}) + N(f_{\frac{3}{4}}) = 3 + 4 + 1 + 2 = 10.$$

5 Orientable fibrations over S^1

Given a fibration $p\colon E \to B$, choose a basepoint $b_0 \in B$ and set $Y = p^{-1}(b_0)$. Let $[Y, Y]$ denote the homotopy classes of homotopy equivalences and define $\tau\colon \pi_1(B, b_0) \to [Y, Y]$ by $\tau[\omega] = \tau_\omega$, the fiber translation obtained from a loop ω representing $[\omega]$. The fibration p is *orientable* if τ is the constant function, that is, the fiber translation τ_ω is homotopic to the identity map for all $[\omega] \in \pi_1(B, b_0)$. The definition is independent of the choice of the regular lifting function that determines τ.

The following product formula is a consequence of Theorem 5.6 of [21].

Proposition 5.1. *Let $p\colon E \to S^1$ be an orientable fibration. If $f\colon E \to E$ is a fiber map with induced map $\bar{f}\colon S^1 \to S^1$, then $N(f) = N(\bar{f})N(f_b)$.*

When we consider n-valued fiber maps of such fibrations, the formula fails to hold. For example, for $Y = S^1$ viewed as the unit circle in the complex plane, define a 2-valued fiber map $\Phi\colon S^1 \times S^1 \multimap S^1 \times S^1$ by $\Phi(w, z) = \{(1, z^3), (-1, z^4)\}$, then

$$N(\Phi) = N(f_1) + N(f_{-1}) = 2 + 3 = 5$$

by Theorem 4.2 whereas, since the induced multimap ϕ is constant, $N(\phi) = 2$ by [17], Corollary 7.3.

Fibrations $p\colon E \to B$ and $p'\colon E' \to B$ are *fiber homotopy equivalent* if there exist fiber maps $\theta\colon E \to E'$ and $\zeta\colon E' \to E$ such that $p'\theta = p, p\zeta = p'$ and $\theta\zeta$ and $\zeta\theta$ are fiber homotopic to the identity maps.

Theorem 5.1. *An orientable fibration $p\colon E \to S^1$ with fiber $Y = p^{-1}(b_0)$ is fiber homotopy equivalent to $\pi_{S^1}\colon S^1 \times Y \to S^1$.*

Proof. By the theorem of [8], the fibration is fiber homotopy equivalent to a bundle. Since orientability is preserved by fiber homotopy equivalence, we may assume without loss of generality that $p\colon E \to B$ is an orientable bundle, with fiber Y. Let $S^1 = c_+ \cup c_-$ where S^1 is the unit circle in the complex plane and c_+ and c_- are the intersections of S^1 with the closed upper and lower half-planes, respectively. Since c_+ and c_- are contractible, by Corollary 11.6 on page 53 of [20] there are fiber-preserving homeomorphisms $h_+\colon c_+ \times Y \to p^{-1}(c_+)$ and $h_-\colon c_- \times Y \to p^{-1}(c_-)$. Let $c_+ \cap c_- = \{z_0, z_1\} = S^0$ and orient c_+ and c_- from z_0 to z_1. For $\epsilon = +, -$ and $v = 0, 1$, define homeomorphisms $h_{\epsilon,v}\colon Y \to p^{-1}(z_v)$ by $h_{\epsilon,v}(y) = h_\epsilon(z_v.y)$ and also set $\mu_v = h_{-,v}^{-1}h_{+,v}\colon Y \to Y$. Since c_+ and c_- contract to z_1, we may use Theorem 2.8.10 of [19] to homotope $h_{-,1}^{-1}$ and $h_{+,1}$ so that μ_1 is homotopic to the identity map, see page 102 of [19].

We define a lifting function $\lambda\colon \Lambda(p) \to E^I$ as follows. Again let $\eta\colon \mathbf{R} \to S^1$ be the covering space. Given $(e_0, \omega) \in \Lambda(p)$ then $\omega(0) = \eta(t_0)$ for some $0 \leq t_0 < 1$. Let $\tilde\omega$ be the lift of ω to $\mathbf{R}$ at t_0 and define $\tilde\omega'\colon I \to \mathbf{R}$ by $\tilde\omega'(s) = (1-s)t_0 + s\tilde\omega(1)$, then $\tilde\omega'$ is homotopic to $\tilde\omega$ by a homotopy L keeping t_0 and $\tilde\omega(1)$ fixed. The path $\omega' = \eta\tilde\omega'$ is homotopic to ω by ηL and it is a union of oriented arcs: $\omega' = \alpha_0 \cdot \alpha_1 \cdot \alpha_2 \cdots \alpha_m$ where $\alpha_j \subseteq c_{\epsilon_j}$ for $\epsilon_j = +$ or $\epsilon_j = -$ and $\alpha_j \cap \alpha_{j+1} \in S^0$. Lift α_0 to E by sending it to $h_{\epsilon_0}(\alpha_0 \times \pi_Y h_{\epsilon_0}^{-1}(e_0))$. Let $e_1 \neq e_0$ be the point in the lift of α_0 that lies in $p^{-1}(S^0)$ and lift α_1 to $h_{\epsilon_1}(\alpha_1 \times \pi_Y h_{\epsilon_1}^{-1}(e_1))$. In general, lift α_j to E by sending it to $h_{\epsilon_j}(\alpha_j \times \pi_Y h_{\epsilon_j}^{-1}(e_j))$, where $e_j \neq e_{j-1}$ is the point in the lift of α_{j-1} that lies in $p^{-1}(S^0)$. This construction defines the path $\lambda(e_0, \omega')$. By the covering homotopy property, we lift the homotopy ηL to $\lambda(e_0, \omega')$ and the lifted homotopy determines $\lambda(e_0, \omega)$.

Let $\omega = \alpha_0\alpha_1 = c_+c_-^{-1}$ and let $e_0 \in p^{-1}(z_0)$. Then

$$e_1 = h_+(z_1, \pi_Y h_+^{-1}(e_0))$$

and

$$\lambda(e_0, \omega) = h_+(c_+ \times \pi_Y h_+^{-1}(e_0)) \cdot h_-(c_-^{-1} \times \pi_Y h_-^{-1}(e_1)).$$

Therefore,

$$\tau_\omega(e_0) = \tau_\omega(h_+(z_0, y_0)) = h_-(z_0, \pi_Y h_-^{-1}h_+(z_1, y_0)) = h_{-,0}(h_{-,1}^{-1}h_{+,1}(y))$$

and so

$$\pi_Y h_-^{-1}\tau_\omega h_{+,0}(y) = h_{-,1}^{-1}h_{+,1}(y) = \mu_1(y).$$

Since the fiber translation τ_ω is homotopic to the identity by the orientability assumption, then μ_1 is homotopic to

$$\pi_Y h_-^{-1} h_{+,0} = h_{-,0}^{-1} h_{+,0} = \mu_0$$

and we conclude that μ_0 is homotopic to the identity because μ_1 is. Therefore the restrictions of h_+ and h_- to $S^0 \times Y$ are homotopic and we let $h_t\colon S^0 \times Y \to p^{-1}(S^0)$ be a homotopy such that $h_0(z_v, y) = h_{-,v}(y)$ and $h_1(z_v, y) = h_{+,v}(y)$. Define

$$H\colon (S^0 \times Y \times I) \cup (c_- \times Y \times \{0\}) \to E$$

by

$$H(z,y,t) = \begin{cases} h_t(z,y) & \text{if} \quad z \in S^0 \\ h_-(z,y) & \text{if} \quad t = 0 \end{cases}$$

and extend H to $H\colon c_- \times Y \times I \to E$ by the fiber homotopy extension theorem [1]. Define $h'_-\colon c_- \times Y \to E$ by $h'_-(z,y) = H(z,y,1)$. Finally, define $h\colon S^1 \times Y \to E$ by

$$h(z,y) = \begin{cases} h_+(z,t) & \text{if} \quad z \in c_+ \\ h'_-(z,t) & \text{if} \quad z \in c_- \end{cases}$$

which is a well-defined fiber map because $h'_-(z_v, y) = h_+(z_v, y)$ for $v = 0, 1$. Since the restriction of h_+ to $z \times Y$ for any $z \in c_+$ is a homotopy equivalence, [7] implies that h is a fiber homotopy equivalence. □

Lemma 5.1. *Let $p\colon E \to S^1$ be an orientable fibration and let $\Phi\colon E \multimap E$ be an n-valued fiber map. Let b_0 and b_1 be fixed points of the induced multimap $\phi\colon S^1 \multimap S^1$ so Φ induces maps $f_{b_v}\colon p^{-1}(b_v) \to p^{-1}(b_v)$ for $v = 0, 1$. If there is a path in the graph*

$$\Gamma_\phi = \{(b, b') \in S^1 \times S^1 \colon b' \in \phi(b)\}$$

from (b_0, b_0) to (b_1, b_1), then $N(f_{b_0}) = N(f_{b_1})$.

Proof. By Theorem 5.1, there are homotopy equivalences $\theta\colon S^1 \times Y \to E$ and $\zeta\colon E \to S^1 \times Y$ such that $p\theta = \pi_{S^1}, \pi_{S^1}\zeta = p$ and $\theta\zeta$ and $\zeta\theta$ are fiber homotopic to the identity maps. Define a multivalued function $\Psi\colon S^1 \times Y \multimap S^1 \times Y$ as follows: for $(b, y) \in S^1 \times Y$ and $e = \theta(b, y)$, if $\Phi(e) = \{e_1, e_2, \ldots, e_n\}$, then $\Psi(b, y) = \{\zeta(e_1), \zeta(e_2), \ldots, \zeta(e_n)\}$. By Theorems 1 and 1′ on page 113 of [3], Ψ is continuous, so Ψ is an n-valued fiber map and its induced multimap is ϕ. For $v = 0, 1$, let $g_{b_v}\colon \{b_v\} \times Y \to \{b_v\} \times Y$ be the restriction of Ψ. Let $\theta_{b_v}\colon \{b_v\} \times Y \to p^{-1}(b_v)$ and $\zeta_{b_v}\colon p^{-1}(b_v) \to \{b_v\} \times Y$

be the restrictions of θ and ζ respectively, then θ_{b_v} is a homotopy equivalence with homotopy inverse ζ_{b_v}. Since $g_{b_v} = \theta_{b_v} f_{b_v} \zeta_{b_v}$, Theorem 5.4 of [14] implies that $N(g_{b_v}) = N(f_{b_v})$. To prove the theorem, we will show that $N(g_{b_0}) = N(g_{b_1})$.

Define a function $\widehat{\Psi}\colon Y \times \Gamma_\phi \to S^1 \times Y$ as follows: for $(y,(b,b')) \in Y \times \Gamma_\phi$, set

$$\widehat{\Psi}(y,(b,b')) = \Psi(b,y) \cap p^{-1}(b').$$

To prove $\widehat{\Psi}$ continuous at $(y,(b,b')) \in Y \times \Gamma_\phi$, write $\widehat{\Psi}(y,(b,b')) = (b',y')$ and choose a neighborhood of (b',y') which we may assume to be of the form $U \times W$ where U is open in $S^1 \subseteq \mathbf{R}^2$ and W is open in Y. Let

$$\gamma(\phi) = \inf\{|b_j - b_k| \colon b_j, b_k \in \phi(b), b \in S^1, b_j \neq b_k\},$$

then $\gamma(\phi) > 0$ because S^1 is compact ([17], page 211). We will find a neighborhood of $(y,(b,b'))$ that is mapped by $\widehat{\Psi}$ into $u \times W$ where u is a neighborhood of b' in U of diameter less than $\gamma(\phi)$. Since $\Psi\colon S^1 \times Y \multimap S^1 \times Y$ is lower semi-continuous, there is a neighborhood of (b,y) in $S^1 \times Y$, which we may assume is of the form $V \times \mathcal{O}$ where V is open in S^1 and $\mathcal{O}$ open in Y, such that $(\bar{b},\bar{y}) \in V \times \mathcal{O}$ implies $\Psi(\bar{b},\bar{y}) \cap (u \times W) \neq \emptyset$. Since u is of diameter less than $\gamma(\phi)$, it must be that $\Psi(\bar{b},\bar{y}) \cap (u \times W)$ is a single point of $S^1 \times Y$. The multimap ϕ is lower semi-continuous so there is a neighborhood $\mathcal{N}$ of b such that $\bar{b} \in \mathcal{N}$ implies $\phi(\bar{b}) \cap u \neq \emptyset$, and the intersection must be a single point of B. Let $\mathcal{M} \subseteq \Gamma_\phi$ be the open subset consisting of all points $(\bar{b},\bar{b}')$ such that $\bar{b} \in \mathcal{N} \cap V$ and $\bar{b}' = \phi(\bar{b}) \cap u$. We claim that $\widehat{\Psi}$ takes the open subset $\mathcal{O} \times \mathcal{M}$ of $Y \times \Gamma_\phi$ into $u \times W$. To prove it, let $\bar{y} \in \mathcal{O}$ and $(\bar{b},\bar{b}') \in \mathcal{M}$ so $\bar{b} \in \mathcal{N} \cap V$. Since $\bar{b} \in V$ and $\bar{y} \in \mathcal{O}$, then $\Psi(\bar{b},\bar{y}) \cap (u \times W) \neq \emptyset$. On the other hand, $\bar{b} \in \mathcal{N}$ so $\phi(\bar{b}) \cap u = \bar{b}'$ which implies $\Psi(\bar{b},\bar{y}) \subseteq p^{-1}(\bar{b}') \times Y$. Thus the single point of $\Psi(\bar{b},\bar{y})$ that lies in $u \times W$ must be $\widehat{\Psi}(\bar{y},(\bar{b},\bar{b}')) = \Psi(\bar{b},\bar{y}) \cap p^{-1}(\bar{b}')$. We conclude that $\widehat{\Psi}(\mathcal{O} \times \mathcal{M}) \subseteq u \times W \subseteq U \times W$ so $\widehat{\Psi}\colon Y \times \Gamma_\phi \to S^1 \times Y$ is continuous.

By hypothesis, there is a path $a\colon I \to \Gamma_\phi$ such that $a(v) = (b_v, b_v)$ for $v = 0,1$. Define $H\colon Y \times I \to Y$ by

$$H(y,t) = \pi_Y \widehat{\Psi}(y,a(t)).$$

The continuity of H follows from the continuity of $\widehat{\Psi}$ that we just established. For $b \in S^1$, define $i_b\colon Y \to p^{-1}(b) \subseteq S^1 \times Y$ by $i_b(y) =$

(b, y). For $v = 0, 1$ we have

$$\begin{aligned} H(y, v) &= \pi_Y \widehat{\Psi}(y, (b_v, b_v)) \\ &= \pi_Y(\Psi(b_v, y) \cap p^{-1}(b_v)) \\ &= \pi_Y g_{b_v} i_{b_v}(y). \end{aligned}$$

Thus the maps $\pi_Y g_{b_0} i_{b_0}, \pi_Y g_{b_1} i_{b_1} \colon Y \to Y$ are homotopic by H so $N(\pi_Y f_{b_0} i_{b_0}) = N(\pi_Y f_{b_1} i_{b_1})$. Since i_{b_j} and the restriction of π_Y to $p^{-1}(b_j)$ are homeomorphisms, $N(g_{b_j}) = N(\pi_Y g_{b_j} i_{b_j})$ by Theorem 5.4 of [14] and we conclude that $N(g_{b_0}) = N(g_{b_1})$. □

Lemma 5.2. *If $\phi, \psi \colon X \multimap Y$ are homotopic n-valued multimaps, then their graphs Γ_ϕ and Γ_ψ are the same homotopy type and thus, in particular, they have the same number of path components.*

Proof. By hypothesis, there is an n-valued multimap $\Delta \colon X \times I \multimap Y$ such that $\Delta(x, 0) = \phi(x)$ and $\Delta(x, 1) = \psi(x)$ for all $x \in X$. The strong deformation retraction $R \colon X \times I \to X \times \{0\}$ induces a strong deformation retraction of covering spaces $\widetilde{R} \colon \Gamma_\Delta \to \Gamma_\phi$ so Γ_Δ and Γ_ϕ are the same homotopy type. The strong deformation retraction of $X \times I$ to $X \times \{1\}$ establishes the same relationship between Γ_Δ and Γ_ψ and completes the proof. □

For n a positive integer, we understand the greatest common divisor of 0 and n to be n.

Proposition 5.2. *Let $\phi \colon S^1 \multimap S^1$ be an n-valued multimap of degree d, then its graph Γ_ϕ has w path components, where w is the greatest common divisor of n and d.*

Proof. For $\eta \colon \mathbf{R} \to S^1$ the covering space, we represent points of S^1 by $\eta(t)$ for $0 \le t < 1$. The n-valued power map $\phi_{n,d} \colon S^1 \multimap S^1$ is defined by

$$\phi_{n,d}(\eta(t)) = \{\eta(\frac{d}{n}t), \eta(\frac{d}{n}t + \frac{1}{n}), \ldots, \eta(\frac{d}{n}t + (n-1)\frac{1}{n})\}.$$

We will first prove that if n and $d \neq 0$ are relatively prime, then $\Gamma_{\phi_{n,d}}$ is path connected. Define $P \colon \mathbf{R}^2 \to S^1 \times S^1$ by $P(x, y) = (\eta(x), \eta(y))$. Let

$$G_{n,d} = \{(x, \frac{d}{n}x) \in \mathbf{R}^2 \colon 0 \le x < n\}$$

and note that $G_{n,d}$ is path connected. For $(x, y) \in G_{n,d}$, write $x = t + m$ where $0 \le t < 1$ and $0 \le m \le n - 1$ is an integer, then

$$\eta(\frac{d}{n}x) = \eta(\frac{d}{n}t + \frac{dm}{n}) = \eta(\frac{d}{n}t + j\frac{1}{n})$$

where the integer j, with $0 \leq j \leq n-1$, is congruent to dm modulo n. Since $\eta(x) = \eta(t)$, we have shown that $P(G_{n,d}) \subseteq \Gamma_{\phi_{n,d}}$.

Now let $(\eta(t), \eta(\frac{d}{n}t + j\frac{1}{n})) \in \Gamma_{\phi_{n,d}}$ where $0 \leq t < 1$ and $0 \leq j \leq n-1$ is an integer. Since d and n are relatively prime, there exist integers a, b such that $1 = ad + bn$. Let $0 \leq m \leq n-1$ be the integer congruent to ja modulo n and let $x = t + m$. We find that

$$\begin{aligned}\left(\frac{d}{n}t + \frac{dm}{n}\right) - \left(\frac{d}{n}t + j\frac{1}{n}\right) &= \frac{dm - j}{n} \\ &= \frac{dm - j(ad + bn)}{n} \\ &= \frac{d(m - ja) - jbn}{n},\end{aligned}$$

which is an integer because $m - ja$ is divisible by n. Therefore

$$\eta(\frac{d}{n}x) = \eta(\frac{d}{n}t + \frac{dm}{n}) = \eta(\frac{d}{n}t + j\frac{1}{n})$$

and we conclude that $\Gamma_{\phi_{n,d}} = P(G_{n,d})$ so $\Gamma_{\phi_{n,d}}$ is path connected.

Now consider $\phi_{n,d}$ for $n \geq 1$ an integer and d any nonzero integer. The power map is w-split as

$$\phi_{n,d} = \{\phi_{n,d}^{(0)}, \phi_{n,d}^{(1)}, \ldots, \phi_{n,d}^{(w-1)}\},$$

for w the greatest common divisor of n and d, where $\phi_{n,d}^{(k)}$ is the $\frac{n}{w}$-valued multimap defined by

$$\phi_{n,d}^{(k)}(\eta(t)) = \{\eta(\frac{d}{n}t + \frac{k}{n}), \eta(\frac{d}{n}t + \frac{k}{n} + \frac{w}{n}), \ldots, \eta(\frac{d}{n}t + \frac{k}{n} + (\frac{n}{w} - 1)\frac{w}{n})\}$$

for $k = 0, 1, \ldots, w-1$. Since $\Gamma_{\phi_{n,d}}$ is the union of the $\Gamma_{\phi_{n,d}^{(k)}}$, and the $\Gamma_{\phi_{n,d}^{(k)}}$ are disjoint sets that are homeomorphic to each other through rotations of $S^1 \times S^1$, it is sufficient to prove that $\Gamma_{\phi_{n,d}^{(0)}}$ is path connected. Noting that $\phi_{n,d}^{(0)} = \phi_{\frac{n}{w}, \frac{d}{w}}$ and that $\frac{n}{w}$ and $\frac{d}{w}$ are relatively prime, the argument above shows that $\Gamma_{\phi_{n,d}^{(0)}}$ is path connected and thus $\Gamma_{\phi_{n,d}}$ has w path components. Now let $\phi\colon S^1 \multimap S^1$ be an n-valued multimap of degree $d \neq 0$. Then by Theorem 3.1 of [4], ϕ is homotopic to $\phi_{n,d}$. Therefore, by Lemma 5.2, Γ_ϕ also has w path components.

If $d = 0$, then ϕ is homotopic to $\phi_{n,0}$. Since $\Gamma_{\phi_{n,0}}$ is the product $\{z_0, z_1, \ldots, z_{n-1}\} \times S^1$ where $z_j = \eta(\frac{j}{n})$, Lemma 5.2 implies that Γ_ϕ has $w = n$ path components in this case also. $\square$

Theorem 5.2. *(Semi-Product Formula) Let $p\colon E \to S^1$ be an orientable fibration and let $\Phi\colon E \multimap E$ be an n-valued fiber map with induced multimap $\phi\colon S^1 \multimap S^1$ of degree d. Let $b_1, b_2, \ldots, b_w \in S^1$, where w is the greatest common divisor of n and d, be fixed points in distinct essential fixed point classes of ϕ such that the points (b_j, b_j) are in distinct path components of Γ_ϕ, then*

$$N(\Phi) = \left| \frac{n}{w} - \frac{d}{w} \right| \sum_{j=1}^{w} N(f_{b_j}).$$

Proof. By Proposition 5.2, Γ_ϕ has w path components, so ϕ is w-split by Proposition 2.1 and we may write $\phi = \{\phi_1, \phi_2, \ldots, \phi_w\}$. By Theorem 3.1 of [4], ϕ is homotopic to $\phi_{n,d}$ so, by Proposition 2.2, each ϕ_j is homotopic to some $\phi^{(k)}_{n,d}$, which is homotopic to $\phi_{\frac{n}{w},\frac{d}{w}}$, and therefore, by Theorem 4.1 of [4],

$$N(\phi_j) = N(\phi^{(k)}_{n,d}) = N(\phi_{\frac{n}{w},\frac{d}{w}}) = \left| \frac{n}{w} - \frac{d}{w} \right|$$

for each j. Proposition 2.3 implies that Φ w-splits as $\Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_w\}$, where each Φ_j is an $\frac{n}{w}$-valued fiber map with induced map ϕ_j. By Theorem 4.1,

$$N(\Phi_j) = \sum_{j=1}^{|\frac{n}{w} - \frac{d}{w}|} N(f_{b_j})$$

where $b_1, b_2, \ldots, b_{|\frac{n}{w} - \frac{d}{w}|}$ are fixed points, one in each essential fixed point class of ϕ_j. Since Γ_{ϕ_j} is path connected, $N(f_{b_j}) = N(f_{b_k})$ for all $j, k = 1, 2, \ldots, |\frac{n}{w} - \frac{d}{w}|$ by Lemma 5.1 so $N(\Phi_j) = |\frac{n}{w} - \frac{d}{w}|\, N(f_{b_j})$ where b_j is in any essential fixed point class of ϕ_j. Therefore, by the Remark on page 218 of [17],

$$N(\Phi) = \sum_{j=1}^{w} N(\Phi_j) = \sum_{j=1}^{w} \left| \frac{n}{w} - \frac{d}{w} \right| N(f_{b_j})$$

where the b_j are in distinct essential fixed point classes of ϕ such that (b_j, b_j) are in distinct path components of Γ_ϕ. □

We obtain the following product formula result:

Corollary 5.1. *Let $p\colon E \to S^1$ be an orientable fibration and let $\Phi\colon E \multimap E$ be an n-valued fiber map with induced multimap $\phi\colon S^1 \multimap S^1$ of degree $d \neq 0$. If d is relatively prime to n, then*

$$N(\Phi) = N(\phi)N(f_b) = |n - d|\, N(f_b)$$

where $b \in B$ is in any essential fixed point class of ϕ.

Suppose B and Y are sets, $S = \{x_1, x_2, \dots, x_n\}$ is an unordered subset of n points of B and $y \in Y$, then $S \times y$ will denote the unordered subset $\{(x_1, y), (x_2, y), \dots, (x_n, y)\}$ of $B \times Y$. For $\phi\colon B \multimap B$ an n-valued function and $f\colon Y \to Y$ single-valued, define the *product n-valued function* $\Phi = \phi \times f\colon E = B \times Y \multimap B \times Y$ by $\Phi(b, y) = \phi(b) \times f(y)$. If ϕ and f are continuous, so also is Φ, that is, it is a *product n-valued multimap.* A product n-valued multimap Φ is an n-valued fiber map with respect to the product bundle $\pi_B\colon B \times Y \to B$ with induced map ϕ and $f_b = f$ for all $b \in \phi(b)$. From Theorem 5.2 we also have the product formula result

Corollary 5.2. *Let $\phi \times f\colon S^1 \times Y \multimap S^1 \times Y$ be a product n-valued multimap where ϕ is of degree d, then*

$$N(\phi \times f) = N(\phi)N(f) = |n - d|\, N(f).$$

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