STABILIZERS OF FIXED POINT CLASSES
AND NIELSEN NUMBERS OF \( n \)-VALUED MAPS

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Abstract

The stabilizer of a fixed point class of a map is the fixed subgroup of the induced fundamental group homomorphism based at a point in the class. A theorem of Jiang, Wang and Zhang is used to prove that if a map of a graph satisfies a strong remnant condition, then the stabilizers of all its fixed point classes are trivial. Consequently, if \( \phi_{p,f} \) is the \( n \)-valued lift to a covering space \( p \) of a map \( f \) with strong remnant of a graph, then the Nielsen numbers are related by the equation \( N(\phi_{p,f}) = n \cdot N(f) \). Additional information concerning Nielsen numbers is obtained for \( n \)-valued lifts of maps of graphs with positive Lefschetz numbers and of maps of lens spaces and for extensions of \( n \)-valued maps.

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1 Introduction

The Nielsen fixed point theory of \( n \)-valued maps, that is, upper and lower semicontinuous functions \( \phi: X \to X \) such that \( \phi(x) \) is an unordered set of exactly \( n \) points of \( X \), was initiated by Schirmer in [14], [15], [16]. She defined a Nielsen number \( N(\phi) \) that is a lower bound for the number of fixed points, that is \( x \in \phi(x) \), for all \( n \)-valued maps that are \( n \)-valued homotopic to \( \phi \). The only examples in those papers were of \( n \)-valued maps of the circle. Classes of nontrivial \( n \)-valued maps of tori were studied, for instance, in [3] and [4], but there were no such examples on other spaces prior to [2]. In that paper, a lifting construction defined \( n \)-valued maps on graphs, orientable double covers of nonorientable manifolds, handlebodies, free \( G \)-spaces and nilmanifolds. The purpose of the present paper is
to extend the classes of spaces on which $n$-valued maps are defined and, especially, to use a recent result of Jiang, Wang and Zhang [12] to refine the computations of the Nielsen numbers of the $n$-valued maps of graphs obtained in [2].

For $p: \widetilde{X} \rightarrow X$ a finite covering space, of degree $n$, of a finite polyhedron and $f: X \rightarrow X$ a map, $\phi_{p,f}: \widetilde{X} \rightarrow \widetilde{X}$, the $n$-valued lift of $f$ is defined by $\phi(\tilde{x}) = \{p^{-1}(fp(\tilde{x}))\}$. The relationship between $N(\phi_{p,f})$ and the Nielsen number $N(f)$ of $f$ was established in [2]. It depends on what is called in [12] the “stabilizer” of a fixed point class, which is defined as follows. Let $x_1$ be a fixed point of $f$, let $f_{x_1}: \pi_1(X,x_1) \rightarrow \pi_1(X,x_1)$ be the induced homomorphism and let $Fix(f_{x_1}) = \{\alpha \in \pi_1(X,x_1) : f_{x_1}(\alpha) = \alpha\}$ be the fixed subgroup of $f_{x_1}$. It has long been known, see page 36 of [10], that if $x_2 \in X$ is in the same fixed point class of $f$ as $x_1$, then $Fix(f_{x_1})$ and $Fix(f_{x_2})$ are isomorphic. If the fixed point classes of $f$ are identified with equivalence classes of the lifts of $f$ to the universal covering space of $X$ under conjugation by deck transformations, then $Fix(f_{x_1})$ can be identified as the stabilizer of the equivalence class corresponding to the fixed point class of $x_1$ under the action of conjugation by deck transformations on the set of lifts of $f$. Since the stabilizer is an invariant of the fixed point class $F_{x_1}$ containing $x_1$, in [12] the group $Fix(f_{x_1})$ is called the stabilizer of the fixed point class $F$ and denoted $Stab(F_{x_1})$. The symbol $#(S)$ will mean the cardinality of the finite set $S$. For $x$ in the fixed point class $F$ we denote by $#(p^{-1}(x)/Stab(F))$ the number of orbits of the restriction to $Stab(F)$ of the monodromy action of $\pi_1(X,x)$ on $p^{-1}(x)$.

The relationship between the Nielsen numbers of $f$ and of $\phi_{p,f}$ is the following

**Theorem 1.1.** ([2]) Let $\phi_{p,f}: \widetilde{X} \rightarrow \widetilde{X}$ be the $n$-valued lift of a map $f: X \rightarrow X$ to the covering space $p: \widetilde{X} \rightarrow X$. Let $F_1, \ldots, F_{N(f)}$ be the essential fixed point classes of $f$ and let $x_j$ be a point of $F_j$, then

$$N(\phi_{p,f}) = \sum_{j=1}^{N(f)} #(p^{-1}(x_j)/Stab(F_j)).$$

Thus $N(\phi_{p,f}) \geq N(f)$. If all the stabilizers are trivial, then $N(\phi_{p,f}) = n \cdot N(f)$.

After presenting in Section 2 a more precise statement of the invariance of the stabilizer, in Section 3 we use Theorem 1.1 to calculate the Nielsen number of an $n$-valued map of a lens space that is a lift of a single-valued lens space map.
For $f: X \to X$ a map of a finite graph, the stabilizer of the fixed point class is a free group that is finitely generated [6]. We denote by $\text{rank}(\text{Stab}(F))$ the rank of the stabilizer of a fixed point class $F$, that is, the smallest number of free generators. We will be making use of the following part of the main result of [12] that relates the rank of the stabilizer of a fixed point class to the fixed point index $\text{ind}(F)$ of that class.

**Theorem 1.2.** (Jiang, Wang, Zhang [12]) Suppose $X$ is a connected finite graph and $f: X \to X$ is a map, then

$$\text{ind}(F) \leq 1 - \text{rank}(\text{Stab}(F))$$

for every fixed point class $F$ of $f$.

This result implies the index bound $\text{ind}(F) \leq 1$ of [11], [13] and it also implies that if $\text{ind}(F) = +1$, then $\text{Stab}(F)$ is the trivial group.

We use Theorem 1.2 in Section 4 to prove that for an $n$-valued lift $\phi_{p,f}$ of a map $f$ of a graph with Lefschetz number $L(f) > 0$, the bound $N(\phi_{p,f}) \geq N(f)$ can be improved to

$$N(\phi_{p,f}) \geq (n - 1)L(f) + N(f).$$

In Section 5 we develop a tool that allows us to describe the stabilizers of fixed point classes in terms of homomorphisms of the fundamental group based at a single point rather than basing the fundamental group at a different fixed point for each fixed point class. We use that tool to obtain a condition on a map of a graph that implies that the stabilizers of all the fixed point classes of $f$ are trivial and thus that $N(\phi_{p,f}) = n \cdot N(f)$ holds for any $n$-valued lift of $f$. The required property, called “strong remnant”, was introduced by Hart [8] as a stronger version of the remnant condition of Wagner in [17]. In Section 6 we prove that, like the remnant condition, the condition of strong remnant is satisfied by “most” maps of graphs, in a sense that is made precise there.

Finally, in Section 7, we show that we can extend an $n$-valued map of a finite polyhedron to an $n$-valued map of a polyhedron that deformation retracts to it, with the Nielsen number unchanged. In particular then, the results regarding $n$-valued maps of graphs can be extended to $n$-valued maps of surfaces with boundary and of handlebodies.

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2 Fixed Subgroups and the Basepoint Isomorphism

Let \( G \) be a group and \( h: G \rightarrow G \) an endomorphism. The **fixed subgroup** of \( h \), which we denote by \( \text{Fix}(h) \) is the group of fixed points of \( h \), that is,

\[
\text{Fix}(h) = \{ g \in G : h(g) = g \}.
\]

Let \( X \) be a space and \( f: X \rightarrow X \) a map such that \( f(x_*) = x_* \). Then \( f \) induces the endomorphism \( f_{x_*}: \pi_1(X, x_*) \rightarrow \pi_1(X, x_*) \) of the fundamental group and we call its fixed subgroup the **fixed subgroup of** \( f_{x_*} \), denoted \( \text{Fix}(f_{x_*}) \). Although the group \( \pi_1(X, x_*) \) is independent of the basepoint \( x_* \), up to isomorphism, Proposition 2.2 of [2] illustrates the fact that, in general, the isomorphism class of the group \( \text{Fix}(f_{x_*}) \) depends on the choice of the fixed point \( x_* \) of \( f \).

Let \( X \) be a space, \( c: [0, 1] \rightarrow X \) a path with \( c(0) = x_0 \) and \( c(1) = x_1 \) and denote by \( \beta_c: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1) \) the **basepoint isomorphism** defined by \( \beta_c[w] = [c^{-1}wc] \) where \( c^{-1}(t) = c(1-t) \).

**Proposition 2.1.** Let \( x_0 \) and \( x_1 \) be fixed points of \( f: X \rightarrow X \) and let \( c: [0, 1] \rightarrow X \) be a path such that \( c(0) = x_0 \) and \( c(1) = x_1 \). Then \( \beta_c(\text{Fix}(f_{x_0})) \subseteq \text{Fix}(f_{x_1}) \) if and only if \( [c(fc)^{-1}] \) is in the centralizer of \( \text{Fix}(f_{x_0}) \). Therefore, the restriction of \( \beta_c \) to \( \text{Fix}(f_{x_0}) \) is an isomorphism onto \( \text{Fix}(f_{x_1}) \) if and only if \( [c(fc)^{-1}] \) is in the centralizer of \( \text{Fix}(f_{x_0}) \) and \( [c^{-1}(fc)] \) is in the centralizer of \( \text{Fix}(f_{x_1}) \).

**Proof.** If \( [c(fc)^{-1}] \) is in the centralizer of \( \text{Fix}(f_{x_0}) \) and \( [w] \in \text{Fix}(f_{x_0}) \), then

\[
\begin{align*}
\text{Fix}_{x_0}(\beta_c[w]) &= [(fc)^{-1}(fw)(fc)] = [(fc)^{-1}w(fc)] \\
&= [c^{-1}(fc)^{-1}w(fc)] = [c^{-1}w(c(fc)^{-1})(fc)] \\
&= \beta_c[w].
\end{align*}
\]

Conversely, suppose \( \beta_c(\text{Fix}(f_{x_0})) \subseteq \text{Fix}(f_{x_1}) \) and let \( [w] \in \text{Fix}(f_{x_0}) \). Then

\[
\text{Fix}_{x_0}\beta_c[w] = [(fc)^{-1}(fw)(fc)] = [c^{-1}wc] = \beta_c[w]
\]

and consequently,

\[
egin{align*}
[w] &= \beta_c \circ \text{Fix}_{x_0} \circ \beta_c[w] = [c(fc)^{-1}(fw)(fc)c^{-1}] \\
&= [c(fc)^{-1}w(fc)c^{-1}] = [c(fc)^{-1}]w[(fc)c^{-1}] \\
&= [c(fc)^{-1}][w][c(fc)^{-1}]^{-1}
\end{align*}
\]
so

\[ [c(fc)^{-1}] [w] = [w] [c(fc)^{-1}]. \]

Fixed points \( x_0 \) and \( x_1 \) are in the same fixed point class if and only if there exists a path \( c \) such that \([c(fc)^{-1}] = 1 \in \pi_1(X, x_0)\) so, in that case, the restriction of \( \beta_c \) is an isomorphism between \( Fix(f_{x_0}) \) and \( Fix(f_{x_1}) \). We will follow the terminology and notation of \([12]\) from now on and call the fixed subgroup \( Fix(f_{x_0}) \) the stabilizer \( Stab(F) \) of its fixed point class \( F \).

If \( \pi_1(X, x, x_0) \) is abelian, then by Proposition 2.1 all the groups \( Stab(F) \) are isomorphic. Theorem 1.1 then implies that

\[ N(\phi_{p,f}) = \#(p^{-1}(x)/Stab(F)) \cdot N(f) \]

where \( x \) is any fixed point of \( f \). If, in addition, \( Stab(F) = 1 \) then \( N(\phi_{p,f}) = n \cdot N(f) \).

In particular, if \( f : X \to X \) is a map of the \( r \)-torus and \( A \) is the \( r \times r \) integer matrix determined by the induced fundamental group homomorphism of \( f \), then

\[ N(\phi_{p,f}) = n \cdot |\det(I - A)|. \]

The reason is that if \( N(f) = |\det(I - A)| \neq 0 \) then \( Stab(F) = 1 \) whereas if \( \det(I - A) = 0 \) then \( f \) is homotopic to a fixed point free map and therefore \( \phi_{p,f} \) has the same property so \( N(\phi_{p,f}) = 0 \).

### 3 Lifts of Maps of Lens Spaces

Let \( L(m, n) \) denote a lens space where \( m \) and \( n \) are relatively prime integers, let \( f : L(m, n) \to L(m, n) \) be a map and let \( x_* \) be a fixed point of \( f \). Then \( f \) induces \( f_{x_*} : \pi_1(L(m, n), x_*) \to \pi_1(L(m, n), x_*) \), an endomorphism of the cyclic group of order \( m \). If \( f_{x_*}(1) = k \) where \( k \neq 1 \), then by Example 3 on page 34 of \([10]\), the Nielsen number is \( N(f) = (k - 1, m) \), the greatest common divisor of \( k - 1 \) and \( m \). Let \( F_* \) be the fixed point class of \( f \) that contains \( x_* \), then \( Stab(F_*) = 1 \). Since \( \pi_1(L(m, n), x_*) \) is abelian, Proposition 2.1 implies that \( Stab(F) = 1 \) for all fixed point classes \( F \). If \( d \) divides \( m \), then there is a covering space \( p : L(d, n) \to L(m, n) \) of degree \( m/d \). Let \( \phi_{p,f} : L(d, n) \to L(d, n) \) be the \( m/d \)-valued lift of \( f \), then the Nielsen number of \( \phi_{p,f} \) is

\[ N(\phi_{p,f}) = \frac{m}{d} \cdot (k - 1, m). \]
4 Maps with Positive Lefschetz Numbers

Let \( f : X \to X \) be a map of a (finite) graph. Since the Nielsen theory of single-valued maps is invariant of homotopy type, we take \( X \) to be a wedge \( X = a_1 \vee a_2 \vee \cdots \vee a_m \) of circles at a vertex that we will always denote by \( x_0 \). The circles are oriented so they generate the free group \( \pi_1(X, x_0) \). We homotope \( f \) so that it maps a neighborhood of \( x_0 \) to \( x_0 \). Then, using the simplicial approximation theorem, we further homotope \( f \) so that each circle \( a_i \) is a union of arcs on each of which the restriction of \( f \) takes the endpoints to \( x_0 \) and the interior homeomorphically onto either some \( a_j - \{x_0\} \) or \( a_j^{-1} - \{x_0\} \). The map \( f \) is then said to be in standard form.

\[ \text{Proposition 4.1.} \quad \text{Let} \quad f : X \to X \quad \text{be a map of a graph. For} \quad p : \tilde{X} \to X \quad \text{a covering space of degree} \quad n, \quad \text{let} \quad \phi_{p,f} : \tilde{X} \to \tilde{X} \quad \text{be the} \quad n \quad \text{-valued lift of} \quad f. \quad \text{If the Lefschetz number} \quad L(f) > 0, \quad \text{then} \]
\[ N(\phi_{p,f}) \geq (n - 1)L(f) + N(f). \]

\[ \text{Proof.} \quad \text{We assume that} \quad f \quad \text{is in standard form and let} \quad F_1, \ldots, F_{N(f)} \quad \text{be the essential fixed point classes. Let} \quad r_i(+) \quad \text{denote the number of fixed points in} \quad F_i \quad \text{of index} \quad +1 \quad \text{and} \quad r_i(-) \quad \text{those of index} \quad -1. \quad \text{Then by Theorem 1.2,} \quad \text{ind}(F_i) = r_i(+) - r_i(-) \leq 1. \quad \text{Therefore, since} \]
\[ \sum_{i=1}^{N(f)} \text{ind}(F_i) = L(f), \quad \text{at least} \quad L(f) \quad \text{of the fixed point classes are of index} \quad +1. \quad \text{By Theorem 1.2, if} \quad \text{ind}(F_i) = 1, \quad \text{then} \quad \text{Stab}(F_i) \quad \text{is the trivial group. Therefore} \quad \#(p^{-1}(x_i)/\text{Stab}(F_i)) = n \quad \text{and consequently} \]
\[ \text{Theorem 1.1 implies that} \quad N(\phi_{p,f}) \geq n \cdot L(f) + (N(f) - L(f)) = (n - 1)L(f) + N(f). \]

5 Maps with Strong Remnant

Let \( X \) be a wedge of oriented circles based at \( x_0 \) and \( f : X \to X \) a map in standard form fixing \( x_0 \) and with one fixed point \( x_j \) for each appearance of a generator \( a_{ij} \) or its inverse in \( f(a_{ij}) \). Then \( f \) induces \( f_{x_0} : \pi_1(X, x_0) \to \pi_1(X, x_0) \). As in [17], we write \( f_{x_0}(a_i) = V_j^{\epsilon_j} a_j^{\epsilon_j} \), where \( \epsilon_j \in \{+1, -1\} \) and define the Wagner tails as follows: \( W_j = V_j \) if \( \epsilon_j = +1 \), \( W_j = V_j a_j^{-1} \) if \( \epsilon_j = -1 \), \( \bar{W}_j = \bar{V}_j^{-1} \) if \( \epsilon_j = +1 \) and \( \bar{W}_j = \bar{V}_j^{-1} a_i \) if \( \epsilon_j = -1 \).

The following proposition allows us to study the stabilizers of the fixed point classes of maps of \( X \) as fixed subgroups of endomor-
Proposition 5.1. For a fixed point $x_j$ of $f$ on a circle $a_i$, define $\theta_{x_j}, \overline{\theta}_{x_j} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$ by $\theta_{x_j}[\omega] = \overline{W}_j^{-1} f(\omega) W_j$ and $\overline{\theta}_{x_j}[\omega] = \overline{W}_j^{-1} f(\omega) \overline{W}_j$. The stabilizer of $F_{x_j}$, the fixed point class of $f$ that contains $x_j$, is isomorphic to the fixed subgroups of $\theta_{x_j}$ and $\overline{\theta}_{x_j}$.

Proof. Without loss of generality, we let $x_j = x_1 \in a_1$. Let $\gamma_+$ be the arc in the circle $a_1$ containing $x_1$ from $x_1$ to $x_0$ in the positive direction and $\gamma_-$ in the negative. Let $\beta_+ : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ be the basepoint isomorphism defined by $\beta_+ [w] = [\gamma_+^{-1} w \gamma_+]$. Lemma 1.2 of [17] implies that $f(\gamma_+) = W_1 \gamma_+$, meaning that they are homotopic relative to the endpoints, so

\[ f_{x_1} \beta_+ [w] = [f_{x_1} (\gamma_+^{-1}) f(w) f_{x_1} (\gamma_+)] = [\gamma_+^{-1} W_1^{-1} f(w) W_1 \gamma_+] = \beta_+ \theta_{x_1} [w]. \]

If $[w] \in \text{Fix}(\theta_{x_1})$, then

\[ f_{x_1} \beta_+ [w] = \beta_+ \theta_{x_1} [w] = \beta_+ [w] \]

so $\beta_+ (\text{Fix}(\theta_{x_1})) \subseteq \text{Fix}(f_{x_1})$. Similarly, $\theta_{x_1} \beta_+^{-1} = \beta_+^{-1} f_{x_1}$ and therefore $\beta_+^{-1} (\text{Fix}(f_{x_1})) \subseteq \text{Fix}(\theta_{x_1})$ so these groups are isomorphic. Replacing $\gamma_+$ with $\gamma_-$, we can prove in the same way that the stabilizer of $F_{x_j}$ is isomorphic to the fixed subgroup of $\overline{\theta}_{x_j}$. \qed

Let $G$ be the free group on generators $a_1, \ldots, a_n$, let $h : G \rightarrow G$ be a homomorphism and set $h(a_i) = A_i$. The homomorphism $h$ has remnant if each $A_i$ can be written in the form $A_i = P_i \overline{A}_i S_i$ where $P_i$ is the longest initial subword of $A_i$ that can be cancelled by $A_i^\epsilon$, for $\epsilon \in \{+1, -1\}$, except for $A_i^{-1}$, the subword $S_i$ is the longest such terminal subword and $\overline{A}_i \neq 1$. Then $\overline{A}_i$ is called the remnant of $A_i$. The homomorphism $h$ has strong remnant if $h$ has remnant and $\overline{A}_i \neq a_i$ for all $i$.

A map $f : (X, x_0) \rightarrow (X, x_0)$ of a graph, in standard form, has remnant [17] if the induced homomorphism $f_{x_0} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$ has remnant and it has strong remnant [8] if $f_{x_0}$ has strong remnant.

Let $|Q|$ denote the length of the word $Q$ in a free group, that is, the minimum number of generators and their inverses needed to write it.

Lemma 5.1. If a homomorphism $h : G \rightarrow G$ has strong remnant, then $\text{Fix}(h) = 1$. 

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Proof. Suppose $h(Q) = Q \neq 1$. Write $Q = \prod_{j=1}^{m} a_{\delta_{j}}^{u_{j}}$ in reduced form for some $m$, then $Q = h(Q) = \prod_{j=1}^{m} A_{\delta_{j}}^{u_{j}}$. The homomorphism $h$ has remnant so $Q$ contains $\sum_{k \in C_{i}} u_{k}$ appearances of the remnant $A_{i}$, where $C_{i} = \{ j | \delta_{j} = i \}$. Since $|Q| = |h(Q)|$, then $Q = \prod_{j=1}^{m} A_{\delta_{j}}^{u_{j}}$ in reduced form. By the uniqueness of the representation of freely reduced words, $X_{i} = a_{i}$ for some $1 \leq i \leq n$. However, this contradicts the hypothesis that $h$ has strong remnant so $\text{Fix}(h) = 1$. \hfill \Box

Let $f : X \to X$ be a map that has remnant and let $x_{i}$ be a fixed point of $f$ corresponding to $a_{i}$. Then, $x_{i}$ is called front-special if $x_{i}$ corresponds to the first letter of the remnant $X_{i}$ and the fixed point index $\text{ind}(f, x_{i}) = -1$. The fixed point $x_{i}$ is called back-special if $x_{i}$ corresponds to the last letter of $X_{i}$ and $\text{ind}(f, x_{i}) = -1$.

**Lemma 5.2.** Suppose $f : X \to X$ has strong remnant. Let $x_{j} \in a_{i}$ be a fixed point that corresponds to $a_{i}$ or $a_{i}^{-1}$ and is not a back-special fixed point of $f$. If $|W_{j}| < |P_{1}A_{1}|$, then the fixed subgroup of $\theta_{x_{j}}$ is trivial and therefore the fixed point class of $f$ containing $x_{j}$ has trivial stabilizer.

**Proof.** Without loss of generality, set $i = 1$. By Proposition 5.1 and Lemma 5.1, it is sufficient to prove that $\theta_{x_{j}}$ has strong remnant. For $1 \leq k \leq n$, let $\theta_{x_{j}}(a_{k}) = W_{j}^{-1}(f_{x_{0}}(a_{k}))W_{j} = Y_{k}$, then since $f$ has remnant, we may write

\[
Y_{k} = W_{j}^{-1}A_{k}W_{j} = W_{j}^{-1}P_{k}A_{k}S_{k}W_{j}.
\]

Since $|W_{j}| < |P_{1}A_{1}|$, then $A_{1}$ cannot be cancelled completely when we reduce $W_{j}^{-1}A_{1}$. Let $Z$ be the subword of $A_{1}$ that is not cancelled, then $Z$ is a subword of $Y_{1}$ because $P_{1}A_{1}$ is not cancelled in $A_{1}W_{j}$ and $|W_{j}| < |P_{1}A_{1}|$. Since $A_{1}$ is the remnant for $A_{1}$, then $Z$ is not cancelled when we reduce $Y_{1} = W_{j}^{-1}A_{1}A_{1}W_{j}$. Similarly, $Z$ is not cancelled when we reduce $Y_{1}Y_{2}, Y_{1}Y_{2}Y_{1}, Y_{1}^{-1}Y_{2}$ and $Y_{1}Y_{2}^{-1}$ for $k = 2, \ldots, n$. Thus the remnant $\overline{Y}_{1}$ of $Y_{1}$ contains $Z \neq 1$. However, $\overline{Y}_{1} = Z = a_{1}$ would imply that $x_{j}$ is a back-special fixed point of $f$, contrary to the hypotheses. For $k = 2, \ldots, n$, since $Y_{k} = W_{j}^{-1}A_{k}W_{j}$ where $W_{j}^{-1}A_{k} = V_{j}^{-1}A_{k}$ or $a_{1}^{-1}V_{j}^{-1}A_{k}$ whereas $A_{k}W_{j} = A_{k}V_{j}$ or $A_{k}V_{j}a_{1}^{-1}$ and $V_{j}$ is a subword of $A_{j}$, then the remnant $A_{k}$ is a subword of $\overline{Y}_{k}$ and, since $f$ has strong remnant, then $\overline{Y}_{k} \neq a_{k}$. Therefore, $\theta_{x_{j}}$ has strong remnant. \hfill \Box

The corresponding argument establishes

**Lemma 5.3.** Suppose $f : X \to X$ has strong remnant, a fixed point $x_{j}$ in the circle $a_{i}$ for some $1 \leq i \leq n$, is not a front-special fixed
point of \( f \) that corresponds to \( a_i \) or \( a_i^{-1} \) and \( |\bar{W}_j| < |X_iS_i| \), then the fixed subgroup of \( \bar{\theta}_{x_j} \) is trivial, where \( \bar{\theta}_{x_j}[\omega] = \bar{W}_j^{-1}[f(\omega)]\bar{W}_j \), and therefore the fixed point class \( F_{x_j} \) of \( f \) containing \( x_j \) has trivial stabilizer.

**Theorem 5.1.** If \( f : X \to X \) has strong remnant, then \( \text{Stab}(F) = 1 \) for any fixed point class \( F \).

**Proof.** Let \( x_j \) be a fixed point of index \(-1\) which, without loss of generality, we may assume is in the circle \( a_1 \). Since \( f \) has strong remnant, a fixed point with index \(-1\) cannot be both front-special and back-special because that would imply that \( X_1 = a_1 \). Since \( \text{ind}(f, x_j) = -1 \), then \( x_j \) is represented by \( a_1 \) so \( W_j = V_j \) and \( W_j = V_j^{-1} \). Therefore at least one of \( |W_j| < |X_1S_1| \) or \( |W_j| < |P_1X_1| \) must be true. Thus, by Lemma 5.2 or 5.3, the stabilizer of the fixed point class containing any points of index \(-1\) is trivial. Since \( \text{ind}(F) \leq 1 \) for any fixed point class \( F \) by Theorem 1.2, a fixed point class containing only points of index \(+1\) consists of a single point. The index of that class is \(+1\) so its stabilizer is trivial by Theorem 1.2.

The example of \( f_{x_0}(a) = a \) on the circle demonstrates that the conclusion of Theorem 5.1 fails if \( f \) does not have strong remnant.

Let \( f : X \to X \) be a map where \( X \) is a graph. For \( p : \tilde{X} \to X \) a covering space of degree \( n \), let \( \phi_{p,f} : \tilde{X} \to \tilde{X} \) be the \( n \)-valued lift of \( f \). As a consequence of Theorem 5.1, if \( f \) has strong remnant then

\[
N(\phi_{p,f}) = n \cdot N(f).
\]

In this case, it is easy to calculate \( N(f) \) because, by part 2 of Theorem 3.3 of [7], if \( f \) has strong remnant then any two fixed points in the same fixed point class are directly related in the sense of [17] so the Nielsen number can be determined by comparing Wagner tails.

6 **Generic Properties of Maps of Graphs**

Let \( f : Y \to Y \) be a map of a graph that is of the homotopy type of a wedge of \( n \) circles then, up to homotopy, \( f \) is characterized by an ordered \( n \)-tuple \( X = (X_1, \ldots, X_n) \) of words in the free group \( F_n \) on \( n \) generators. Let \( B^n(M) \) denote the set of \( n \)-tuples of words in \( F_n \) all of length less than or equal to \( M \) and \( B^n(m, M) \subseteq B^n(M) \) the \( n \)-tuples for which all the words are of length at least \( m \). For a
property possessed by a subset of \( n \)-tuples, we identify the property with the set itself. Specializing a concept due to Gromov [5], we define a property \( S \) of \( n \)-tuples of words in \( F_n \) to be \textit{generic} if

\[
\lim_{k \to \infty} \frac{\#(S \cap B^n(k))}{\#(B^n(k))} = 1.
\]

We denote by \( S(r) \), for \( r \geq 1 \), the set of \( n \)-tuples \( X = (X_1, \ldots, X_n) \) that have \textit{minimum remnant length} \( r \), that is, \( |X_i| \geq r \) for all \( i \), compare [7]. Thus \( S(1) \) is the set of \( n \)-tuples with remnant in the sense of [17]. The following result is a consequence of [1]. However, we take this opportunity to present a self-contained, elementary proof, modelled on that of Theorem 3.7 of [17], in order to add some details and to correct some minor errors in the published proof of Theorem 3.7.

**Theorem 6.1.** The \textit{minimum remnant length property} \( S(r) \) is \textit{generic} for all \( r \geq 1 \).

**Proof.** Suppose given \( \epsilon > 0 \). We write

\[
\frac{\#(S(r) \cap B^n(k))}{\#(B^n(k))} = \frac{\#(S(r) \cap B^n(k))}{\#(B^n(m_0, k))} \cdot \frac{\#(B^n(m_0, k))}{\#(B^n(k))}.
\]

We will prove that there exists \( m_0 \) such that there is \( M > m_0 \) with the property that if \( k \geq M \) then each factor of the product is greater than \( \sqrt{1 - \epsilon} \).

Without loss of generality, we assume that \( m \) has the same parity as \( r \) so that \( (m - r)/2 \) is an integer. If \( X \in B^n(m, M) \) does not have minimum remnant length \( r \), there is at least one \( X_i \) such that at least one of \( |P_i| > (m - r)/2 \) or \( |S_i| > (m - r)/2 \) is true. We observe that

\[
\frac{\#(X = P Y; X \in B^1(m, M))}{\#(B^1(m, M))} \leq \frac{1}{(2n)(2n - 1)^{(m-r)/2-1}} < \frac{1}{(2n - 1)^{(m-r)/2}}
\]

and that this inequality does not depend on the length of the word \( X_i \) and so it holds for any value of \( M > m \). Since \( X_i \) must be tested against all \( X_j^{\pm 1} \in X \) except \( X_i^{-1} \) at both the start and the end of the word,

\[
\frac{\#((X_i \notin S(r)) \cap B^1(m, M))}{\#(B^1(m, M))} \leq \frac{2(2n - 1)}{(2n - 1)^{(m-r)/2}}
\]

\[10\]
and therefore
\[ \frac{\#(\{ x \notin S(r) \cap B^n(m, M) \})}{\#(B^n(m, M))} \leq \frac{2n(2n - 1)}{(2n - 1)^{(m-r)/2}} \leq \frac{4n^2}{(2n - 1)^{(m-r)/2}}. \]

We choose \( m = m_0 \) so that
\[ \frac{4n^2}{(2n - 1)^{(m_0-r)/2}} < 1 - \sqrt{1 - \epsilon}. \]

Denoting the negation of a property \( S \) by the symbol \( \sim S \), we have proved that
\[ \frac{\#(\sim S(r) \cap B^n(m_0, M))}{\#(B^n(m_0, M))} < \frac{4n^2}{(2n - 1)^{(m_0-r)/2}} \]
and therefore that
\[ \frac{\#(S(r) \cap B^n(m_0, M))}{\#(B^n(m_0, M))} > \sqrt{1 - \epsilon}. \]

Now, choose \( M \) so that
\[ \frac{1 - (2n - 1)^{m_0}}{1 - (2n - 1)^M} < 1 - \sqrt{1 - \epsilon}. \]

then for \( k \geq M \) we have
\[
\frac{\#(B^n(m_0, k))}{\#(B^n(k))} = \frac{\sum_{j=m_0}^{k} 2n(2n - 1)^{j-1}}{1 + \sum_{j=1}^{k} 2n(2n - 1)^{j-1}} > \frac{\sum_{j=m_0}^{k} 2n(2n - 1)^{j-1}}{\sum_{j=1}^{k_0} 2n(2n - 1)^{j-1}} = \frac{\sum_{j=1}^{k} (2n - 1)^{j-1} - \sum_{j=1}^{m_0-1} (2n - 1)^{j-1}}{\sum_{j=1}^{m_0} (2n - 1)^{j-1}} = 1 - \frac{\sum_{j=1}^{m_0-1} (2n - 1)^{j-1}}{\sum_{j=1}^{k} (2n - 1)^{j-1}} = 1 - \frac{1 - (2n - 1)^{m_0}}{1 - (2n - 1)^k} \geq \sqrt{1 - \epsilon}.
\]

Therefore
\[ \frac{\#(S(r) \cap B^n(k))}{\#(B^n(k))} > 1 - \epsilon \]
and we have proved that
\[ \lim_{k \to \infty} \frac{\#(S(r) \cap B^n(k))}{\#(B^n(k))} = 1. \]
We denote the set of \( n \)-tuples that have strong remnant by \( S(s) \) then, since \( S(2) \subset S(s) \), we have

**Corollary 6.1.** The strong remnant property \( S(s) \) is generic.

In [9], a map \( f \) is called essentially fix trivial if \( \text{Stab}(F) = 1 \) for all essential fixed point classes \( F \) of \( f \). We extend the definition by calling \( f \) totally fix trivial if \( \text{Stab}(F) = 1 \) for all its fixed point classes, essential or not. Thus Theorem 6.1 implies that “most” maps of wedges of circles have only trivial stabilizers of their fixed point classes in the following sense:

**Corollary 6.2.** For maps of graphs, the totally fix trivial property is generic.

Therefore, for “most” \( n \)-valued lifts \( \phi_{p,f} \) of maps \( f \) of graphs, the Nielsen number is \( N(\phi_{p,f}) = n \cdot N(f) \).

## 7 Extensions of \( n \)-Valued Maps

Let \( X \) be a finite polyhedron, \( Y \) a subpolyhedron of \( X \), \( \phi: Y \to Y \) an \( n \)-valued map, and \( r: X \to Y \) a strong deformation retraction to the subpolyhedron. Then the \( n \)-valued map \( \hat{\phi} = i \circ \phi \circ r: X \to X \), where \( i: Y \to X \) is inclusion, is well-defined. We call \( \hat{\phi} \) the extension of \( \phi \) with respect to the retraction \( r \).

**Theorem 7.1.** The Nielsen number of an \( n \)-valued map is the same as that of any extension of it, that is, \( N(\hat{\phi}) = N(\phi) \).

**Proof.** By Theorem 6 of [14], we can homotope \( \phi \) so that the fixed point set \( \text{Fix}(\phi) \) is finite. Note that \( \text{Fix}(\hat{\phi}) = \text{Fix}(\hat{\phi}|Y) = \text{Fix}(\phi) \).

We claim that the fixed point classes of \( \phi \) and \( \hat{\phi} \) are identical. Suppose \( x \) and \( y \) are in the same fixed point class of \( \phi \) in the sense of [15]. That means that there exists a path \( c: [0,1] \to Y \) from \( x \) to \( y \) such that \( g_j \), for some \( j \), is homotopic to \( c \) relative to the endpoints, where \( \{g_i\}_{1 \leq i \leq n} \) is the splitting of \( \phi c: I \to Y \). Since the path \( c \) is in \( Y \), then \( \hat{\phi} \circ c = \phi \circ c \). Thus, they have the same splitting and \( g_i \) is a map in the splitting of \( \hat{\phi} \circ c \). Therefore, \( x \) and \( y \) are in the same fixed point class of \( \hat{\phi} \). Conversely, suppose \( x \) and \( y \) are in the same fixed point class of \( \hat{\phi} \). Then, there exists a path \( \hat{c}: [0,1] \to X \) from \( x \) to \( y \) such that \( \hat{g}_j \) is homotopic to \( c \), for some integer \( j \), where \( \{\hat{g}_i\}_{1 \leq i \leq n} \) is the splitting of \( \hat{\phi} \circ c \). Therefore, \( r \hat{g}_j \) and \( rc \) are homotopic relative to the endpoints, which are in \( Y \). Since the image of \( \hat{\phi} \) is in \( Y \), the map \( r \circ \hat{g}_j = g_j \) is a member of the splitting of \( \phi \circ r \circ c \). Thus, \( r \circ c \)
is a path from $x$ to $y$ such that $r \circ c$ is homotopic to $g_j$ relative to the endpoints. This shows that $x$ and $y$ are in the same fixed point class of $\phi$ and this establishes the claim that the fixed point classes of $\phi$ and of $\hat{\phi}$ are identical.

Since $Y$ is locally contractible, there is a contractible (open) neighborhood $U_0$ of a fixed point $y_0$ of $\phi$. Therefore, there exist maps $\{f_i: U_0 \to Y\}_{1 \leq i \leq n}$ splitting $\phi$ such that $f_j(y_0) = y_0$ for some $j$. If $x \in r^{-1}(U_0) = U$, then

$$\hat{\phi}(x) = \phi(r(x)) = \{f_i(r(x))\}_{1 \leq i \leq n}$$

and $y_0 = \hat{f}_j(y_0) = f_j(r(y_0))$. Let $V_0 \subset U_0$ be a neighborhood of $y_0$ such that $f_j(V_0) \subset U_0$ and let $V = r^{-1}(V_0)$ so $\hat{f}_j(V) \subset U_0 \subset U$. Consider

$$\hat{f}_j|V: \left( V \xrightarrow{r} V_0 \xrightarrow{f} U_0 \right) \hookrightarrow U$$

and

$$f_j|V_0: V_0 \hookrightarrow \left( V \xrightarrow{r} V_0 \xrightarrow{f} U_0 \right).$$

By the commutativity property of the fixed point index, we have $\text{ind}(\hat{f}_j|V, V) = \text{ind}(f_j|V_0, V_0)$. The excision property implies that $\text{ind}(\hat{f}_j, U) = \text{ind}(f_j, U_0)$. Thus, according to the definition in [15], $\text{ind}(\hat{\phi}, x_0) = \text{ind}(\phi, x_0)$ and therefore $\mathcal{F}$ is essential as a fixed point class of $\phi$ if and only if it is essential as a fixed point class of $\hat{\phi}$ and we conclude that $N(\hat{\phi}) = N(\phi)$.

We may construct a class of multiply fixed $n$-valued maps of surfaces with boundary as follows. Suppose a graph $\tilde{X}$ is a finite covering of degree $n$ of a graph $X$ by a covering map $p: \tilde{X} \to X$. Suppose $Y$ is a surface with boundary containing $\tilde{X}$ and there is a strong deformation retraction $r: Y \to \tilde{X}$. Let $f: X \to X$ be a map and extend the lift $\phi_{p,f}: \tilde{X} \to \tilde{X}$ to $\hat{\phi}_{p,f} = \iota \circ \phi_{p,f} \circ r: Y \to Y$. If $f$ has strong remnant, then

$$N(\hat{\phi}_{p,f}) = n \cdot N(f)$$

by Theorem 5.1.

References


