LIFTING CLASSES FOR THE FIXED POINT THEORY OF n-VALUED MAPS

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Abstract

The theory of lifting classes of single-valued maps is extended to n-valued maps by replacing liftings to universal covering spaces by liftings with codomain an orbit configuration space, a structure recently introduced by Xicoténcatl. An equivalence relation, that reduces to conjugation by covering transformations if $n = 1$, is defined on these liftings. The number of equivalence classes of liftings of an n-valued map f is called the conjugacy class number and denoted $C(f)$. The fixed point classes of f are the projections of fixed point sets of these liftings and are the same as those of Schirmer. We relate our liftings to liftings to the universal covering spaces and we show that in many cases they are the same. A twisted conjugacy relation is extended from single-valued to n-valued maps and the number of equivalence classes is called the Reidemeister number of f and denoted $R(f)$, as in the single-valued theory. We prove that it is related to the conjugacy class number by the equation $C(f) = R(f)^n$. The Jiang subgroup is extended to n-valued maps as a subgroup of a semidirect product and used to find conditions under which the Nielsen number of an n-valued map equals its Reidemeister number.

Keywords and Phrases: lifting class, n-valued map, Reidemeister number, Nielsen number, configuration space, universal covering space, conjugacy class number, Jiang subgroup, orbit configuration space, semidirect product, fixed point class, Reidemeister relation, Reidemeister class, cyclic homotopy, braid group

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1 Introduction

Throughout the paper, the space X will be a connected finite polyhedron. Given some natural number $n > 0$, a set-valued function $f : X \multimap X$ is an *n-valued map* if it is a continuous, that is, both upper and lower semi-continuous, and the cardinality of $f(x)$ is exactly n for each x, see [3].

For the Nielsen fixed point theory of a single-valued map $f\colon X \to X$, the set of liftings $\tilde{f}\colon \tilde{X} \to \tilde{X}$ to the universal covering space $p\colon \tilde{X} \to X$ is partitioned into equivalence classes under conjugation by covering transformations. The fixed point sets $\mathrm{Fix}(\tilde{f})$ of equivalent liftings, if nonempty, are mapped by p to the same subsets of $\mathrm{Fix}(f)$. An equivalence class is called a *lifting class* and the number of such classes, which may be infinite, is the *Reidemeister number* $R(f)$ of the map f [12], [13]. The sets $p\,\mathrm{Fix}(\tilde{f})$ are the *fixed point classes* of the map f and the number of such classes of nonzero fixed point index, called the *Nielsen number*, is a lower bound for the number of fixed points of every map homotopic to f.

The purpose of this paper is to extend the theory of lifting classes to the setting of n-valued maps. In order to do so, following [8] we will view an n-valued map as a single-valued map from X to a space of subsets of X. Let $F_n(X)$ be the *configuration space of n ordered points on X*, defined as:

$$F_n(X) = \{(x_1, \ldots, x_n) \mid i \neq j \text{ implies } x_i \neq x_j\}.$$

which is topologized as a subset of the n-fold Cartesian product of X. Let $D_n(X)$ be the *configuration space of n unordered points on X*, defined as:

$$D_n(X) = \{\{x_1, \ldots, x_n\} \mid i \neq j \text{ implies } x_i \neq x_j\}.$$

Thus $D_n(X)$ is the orbit space of $F_n(X)$ under the free action of the symmetric group Σ_n and the quotient map $q : F_n(X) \to D_n(X)$, which induces the quotient topology on $D_n(X)$, is a covering space of order $n!$. We will not distinguish between an n-valued map $f\colon X \multimap X$, and the corresponding function $f\colon X \to D_n(X)$, which is also continuous [3]. Thus we may refer to a map $f\colon X \to D_n(X)$ as an n-valued map.

As we will discuss in Section 2, the lifting classes for $f\colon X \to D_n(X)$ will not be classes of maps of the corresponding universal covering spaces because if $n > 1$ then such a lifting does not have a well-defined fixed point set. Instead we will consider liftings $\bar{f}$ of f from the universal covering space $\tilde{X}$ to the orbit configuration space,

a covering space of $D_n(X)$ for which $\mathrm{Fix}(\bar{f})$ is well-defined. We introduce an equivalence relation on such liftings which reduces to conjugation by covering transformations when $n = 1$. The number of equivalence classes of liftings is called the *conjugacy class number* of f and denoted by $C(f)$.

Helga Schirmer, in initiating the Nielsen fixed point theory for n-valued maps in [14], extended the classical definition of the fixed point classes to n-valued maps. As a model for her definition, she did not use images of fixed point sets of liftings but, instead, an equivalent definition in terms of paths in the space. Gert-Jan Dugardein reformulated Schirmer's theory in terms of a definition of lifting classes different than the one we introduce in Section 2, but one that is equivalent to it, and he showed that the fixed point classes defined as images of the fixed point sets of those liftings are the same as the classes defined by Schirmer. We will present Dugardein's results in Section 3 and demonstrate that our definition of the fixed point classes is equivalent to Schirmer's.

In the fixed point theory of single-valued maps, there is a twisted conjugacy relation on the fundamental group $\pi_1(X)$ of X whose equivalence classes are in one-to-one correspondence with the lifting classes. Section 4 is devoted to a twisted conjugacy relation for n-valued maps, that extends the definition for single-valued maps, that is an equivalence relation on $\pi_1(X)^n \times \{1, \ldots, n\}$, where $\pi_1(X)^n$ is the product of n copies of $\pi_1(X)$. We define the *Reidemeister number* of an n-valued map to be the number of equivalence classes. We prove that the Reidemeister number is related to the conjugacy class number by the equation $C(f) = R(f)^n$.

In Section 5 we calculate the Reidemeister number for all n-valued maps of the circle.

The induced fundamental group homomorphism of a single-valued map may be viewed as the homomorphism of the groups of covering transformations induced by a lifting to the universal covering spaces. In Section 6, we discuss the corresponding induced homomorphism of groups of covering transformations when an n-valued map is lifted to a map from the universal covering space of X to the orbit configuration space. We prove that if X is a manifold of dimension at least three, then that homomorphism corresponds to the induced fundamental group homomorphism.

For a map $f\colon X \to X$, Jiang in [11] introduced a subgroup $J(\tilde{f})$ of the fundamental group, that is called the *Jiang subgroup* of the map f. It consists of the elements $\alpha \in \pi_1(X)$ such that there is a homotopy $H\colon X \times I \to X$ with the property that $H(x, 0) =$

$H(x,1) = f(x)$ and, when lifted to the universal covering space, it induces a homotopy between $\tilde{f}$ and $\alpha\tilde{f}$, where α is identified with the corresponding covering transformation. In Section 7, we extend the Jiang subgroup concept to the setting of n-valued maps and use it to present conditions under which the Nielsen number equals the Reidemeister number.

A final section discusses the previous topics for split n-valued maps, that is, maps $f\colon X \to D_n(X)$ for which there exist single-valued maps $f_1, \ldots, f_n\colon X \to X$ such that $f(x) = \{f_1(x), \ldots, f_n(x)\}$ for all $x \in X$.

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2 Coverings of $D_n(X)$

Let $r : \tilde{F}_n(X) \to F_n(X)$ be the universal covering space. There is a covering $q : F_n(X) \to D_n(X)$ so, since $\tilde{F}_n(X)$ is a universal cover, it is simply connected and thus $\tilde{F}_n(X)$ is the universal covering space of $D_n(X)$ with covering projection $qr : \tilde{F}_n(X) \to D_n(X)$.

Let $p\colon \tilde{X} \to X$ be the universal covering space of X. A map $f\colon X \to D_n(X)$ has a lifting $\tilde{f}$ to the universal covering spaces:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{F}_n(X) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle qr} \\ X & \xrightarrow{f} & D_n(X) \end{array}$$

But, in contrast to the setting of single-valued maps, the lifting of an n-valued map $f : X \to D_n(X)$ to the universal covering spaces, for $n > 1$, is not a convenient object of study because the fixed point set of $\tilde{f}$ is not defined. Consequently, the map f will not be lifted to the universal covering spaces, but instead we will make use of an intermediate covering.

Let E be a space and G be a group acting on E such that the projection $E \to E/G$ is a principal fibration. Then Xicoténcatl defined the *orbit configuration space of n ordered points* (see [16]) as:

$$F_n(E,G) = \{(e_1, \ldots, e_n) \in F_n(E) \mid Ge_i \neq Ge_j \text{ for } i \neq j\}.$$

We will make use of the orbit configuration space $F_n(\tilde{X}, \pi_1(X))$ which we will write more compactly as $F_n(\tilde{X}, \pi)$. In this context, Theorem 2.3 of [16] describes a covering $F_n(\tilde{X}, \pi)$ of $D_n(X)$.

Theorem 2.1 (Xicoténcatl)**.** *There is a covering map*

$$p^n \colon F_n(\tilde{X}, \pi) \to D_n(X),$$

with covering group the semidirect product $\pi_1(X)^n \rtimes \Sigma_n$, where $\pi_1(X)^n$ is the direct product of n copies of the fundamental group and p^n applies $p \colon \tilde{X} \to X$ to each element of an n-element configuration. The action of $\pi_1(X)^n \rtimes \Sigma_n$ on $F_n(\tilde{X}, \pi)$ is given by:[1]

$$(\alpha_1, \ldots, \alpha_n; \sigma) \cdot (\tilde{x}_1, \ldots, \tilde{x}_n) = (\alpha_1 \tilde{x}_{\sigma(1)}, \ldots, \alpha_n \tilde{x}_{\sigma(n)}).$$

The group operation and inverse for the semidirect product take the form:

$$\begin{aligned}(\alpha_1, \ldots, \alpha_n; \sigma)(\beta_1, \ldots, \beta_n; \rho) &= (\alpha_1 \beta_{\sigma(1)}, \ldots, \alpha_n \beta_{\sigma(n)}; \sigma \circ \rho) \\ (\alpha_1, \ldots, \alpha_n; \sigma)^{-1} &= (\alpha^{-1}_{\sigma^{-1}(1)}, \ldots, \alpha^{-1}_{\sigma^{-1}(n)}; \sigma^{-1})\end{aligned}$$

Let $f \colon X \to D_n(X)$ be an n-valued map. We choose basepoints as follows. First select some $\tilde{x}^* \in \tilde{X}$ and let $x^* = p(\tilde{x}^*)$. Set $x^{(0)} = f(x^*) \in D_n(X)$ and choose $\tilde{x}^{(0)} = (\tilde{x}_1^{(0)}, \ldots, \tilde{x}_n^{(0)}) \in F(\tilde{x}, \pi)$ such that

$$x^{(0)} = \{p(\tilde{x}_1^{(0)}), \ldots, p(\tilde{x}_n^{(0)})\}.$$

Let $\bar{f}^* \colon \tilde{X} \to F_n(\tilde{X}, \pi)$ be the lifting of f such that $\bar{f}^*(\tilde{x}^*) = \tilde{x}^{(0)}$. The lifting $\bar{f}^*$ is characterized by the property that it preserves basepoints, so we will call it the *basic lifting* of the n-valued map f. By Theorem 2.1, all liftings of f are of the form $(\alpha; \eta)\bar{f}^*$ for some $(\alpha; \eta) \in \pi_1(X)^n \rtimes \Sigma_n$.

Since $F_n(\tilde{X}, \pi) \subseteq F_n(\tilde{X})$, we may write $\bar{f}^*$ in terms of the coordinate self-maps of $\tilde{X}$ as $\bar{f}^* = (\bar{f}_1^*, \ldots, \bar{f}_n^*)$ which we may order such that $\bar{f}_i^*(\tilde{x}^*) = \tilde{x}_i^{(0)}$. Then we have a diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "(\bar{f}_1^*, \ldots, \bar{f}_n^*)"] \arrow[d, "p"] & F_n(\tilde{X}, \pi) \arrow[d, "p^n"] \\ X \arrow[r, "f"] & D_n(X) \end{tikzcd}$$

Recall from single-valued theory that choosing a lifting $\tilde{f} \colon \tilde{X} \to \tilde{X}$ to the universal covering space of a single-valued function $f \colon X \to$

[1] In [16], the definition of the action uses σ^{-1} rather than σ throughout. This is simply a convention, and using σ is more convenient in this paper.

X, then the liftings are the $\alpha\tilde{f}$ where α is a covering transformation and therefore it may be associated with an element of $\pi_1(X)$. Liftings $\alpha\tilde{f}$ and $\beta\tilde{f}$ are *equivalent via* $\mu \in \pi_1(X)$ if

$$\alpha\tilde{f} = \mu^{-1}\beta\tilde{f}\mu.$$

For an n-valued map $f\colon X \to D_n(X)$ the liftings are the

$$(\alpha;\eta)\bar{f}^* = (\alpha_1\bar{f}^*_{\eta(1)}, \ldots, \alpha_n\bar{f}^*_{\eta(n)})$$

where $(\alpha;\eta) = (\alpha_1, \ldots, \alpha_n; \eta) \in \pi_1(X)^n \rtimes \Sigma_n$ and $\alpha_i\bar{f}^*_{\eta(i)}\colon \tilde{X} \to \tilde{X}$.

Define liftings $(\alpha;\eta)\bar{f}^*$ and $(\beta;\theta)\bar{f}^*$ to be *conjugate via* $\mu = (\mu_1, \ldots, \mu_n) \in \pi_1(X)^n$ if

$$\alpha_i\bar{f}^*_{\eta(i)} = \mu_i^{-1}\beta_i\bar{f}^*_{\theta(i)}\mu_i$$

for each $i = 1, \ldots, n$. Denote the lifting class containing $(\alpha;\eta)\bar{f}^*$ by $[(\alpha;\eta)]$.

Proposition 2.2. *Conjugacy via elements of $\pi_1(X)^n$ is an equivalence relation.*

Proof. For reflexivity, we have $\alpha_i\bar{f}^*_{\eta(i)} = 1^{-1}\alpha_i\bar{f}^*_{\eta(i)}1$.

For symmetry, let $\alpha_i\bar{f}^*_{\eta(i)} = \mu_i^{-1}\beta_i\bar{f}^*_{\theta(i)}\mu_i$. Then composing on the right by μ_i^{-1} and acting on the left by μ_i gives:

$$\mu_i\alpha_i\bar{f}^*_{\eta(i)}\mu_i^{-1} = \beta_i\bar{f}^*_{\theta(i)}$$

and so $\beta_i\bar{f}^*_{\theta(i)}$ is conjugate to $\alpha_i\bar{f}^*_{\eta(i)}$ by μ_i^{-1}.

For transitivity, let $\alpha_i\bar{f}^*_{\eta(i)} = \mu_i^{-1}\beta_i\bar{f}^*_{\theta(i)}\mu_i$ and $\beta_i\bar{f}^*_{\theta(i)} = \nu_i^{-1}\delta_i\bar{f}^*_{\chi(i)}\nu_i$. Then

$$\begin{aligned}\alpha_i\bar{f}^*_{\eta(i)} &= \mu_i^{-1}\nu_i^{-1}\delta_i\bar{f}^*_{\chi(i)}\nu_i\mu_i \\ &= (\nu_i\mu_i)^{-1}\delta_i\bar{f}^*_{\chi(i)}(\nu_i\mu_i).\end{aligned}$$

□

In the fixed point theory of a single-valued map, the number of equivalence classes of liftings to the universal covering space is called its Reidemeister number and denoted $R(f)$. For an n-valued map $f\colon X \to D_n(X)$ the number of equivalence classes of liftings will be called its *conjugation class number* and denoted $C(f)$. We will reserve the term Reidemeister number and the notation $R(f)$ for another concept, for reasons that will become evident later in this paper. As with the Reidemeister number, $C(f)$ is either a natural number or ∞.

It is natural to consider a more restrictive relation: we say that $(\alpha;\eta)\bar{f}^*$ and $(\beta;\theta)\bar{f}^*$ are *uniformly conguate* via $\gamma \in \pi_1(X)$ if:

$$\alpha_i \bar{f}^*_{\eta(i)} = \gamma^{-1}\beta_i \bar{f}^*_{\theta(i)}\gamma$$

for each $i = 1, \ldots, n$. It is straightforward to check that this is also an equivalence relation, and we let $C_u(f)$ denote the number of uniform conjugacy classes. Clearly we have $C(f) \leq C_u(f)$, but as we will demonstrate in Section 5, $C_u(f)$ is very often infinite, in particular it is infinite for all maps on the circle.

Theorem 2.3. *Let $f\colon X \to D_n(X)$ be an n-valued map with basic lifting $\bar{f}^*$. Then*

$$\operatorname{Fix}(f) = \bigcup_{\gamma\in\pi_1(X)} \bigcup_{i=1}^{n} p\operatorname{Fix}(\gamma\bar{f}^*_i)$$

where $\operatorname{Fix}(f) = \{x \in X \mid x \in f(x)\}$.

Proof. Let $\tilde{x} \in \tilde{X}$ such that $\tilde{x} = \gamma\bar{f}^*_i(\tilde{x})$. Then for $x = p(\tilde{x})$, we have

$$x = p(\gamma\bar{f}^*_i(\tilde{x})) \in fp(\tilde{x}) = f(x)$$

and we have proved that

$$\bigcup_{\gamma\in\pi_1(X)} \bigcup_{i=1}^{n} p\operatorname{Fix}(\gamma\bar{f}^*_i) \subseteq \operatorname{Fix}(f)$$

Now suppose $x \in \operatorname{Fix}(f)$, that is, $x \in f(x)$. Let

$$p^n\bar{f}^* = \{p\bar{f}^*_1, \ldots, p\bar{f}^*_n\}\colon \tilde{X} \to D_n(X).$$

Since x is a fixed point, there exists $i \in \{1, \ldots, n\}$ and $\tilde{x} \in p^{-1}(x)$ such that $p\bar{f}^*_i(\tilde{x}) = x$ so $\bar{f}^*_i(\tilde{x}) \in p^{-1}(x)$ and thus there exists $\gamma \in \pi_1(X)$ such that $\gamma\bar{f}^*_i(\tilde{x}) = \tilde{x}$. We have proved that $x \in p\operatorname{Fix}(\gamma\bar{f}^*_i)$ so

$$\operatorname{Fix}(f) \subseteq \bigcup_{\gamma\in\pi_1(X)} \bigcup_{i=1}^{n} p\operatorname{Fix}(\gamma\bar{f}^*_i). \qquad \square$$

For such a map $f\colon X \to D_n(X)$ with its basic lifting $\bar{f}^*$, we define the *fixed point classes of* f to be all sets of the form

$$p\operatorname{Fix}(\gamma\bar{f}^*_i) = \{p(\tilde{x}) \mid \gamma\bar{f}^*_i(\tilde{x}) = \tilde{x}\} \subseteq \operatorname{Fix}(f)$$

where $\gamma \in \pi_1(X)$. We note that $\operatorname{Fix}(\gamma\bar{f}^*_i)$ may be empty and therefore a fixed point class $p\operatorname{Fix}(\gamma\tilde{f}_i)$ may be the empty set.

Lemma 2.4. *Let $f : X \to D_n(X)$ be an n-valued map, and $\bar{f}^* = (\bar{f}_1^*, \ldots, \bar{f}_n^*)$ be the basic lifting of f. There is a function $\sigma : \pi_1(X) \to \Sigma_n$ and, for each $i = 1, \ldots, n$, a function $\phi_i : \pi_1(X) \to \pi_1(X)$ such that*

$$\bar{f}_i^*(\gamma\tilde{x}) = \phi_i(\gamma)\bar{f}^*_{\sigma_\gamma(i)}(\tilde{x}) \tag{1}$$

for all $\tilde{x} \in \tilde{X}$, where σ_γ denotes the image of $\gamma \in \pi_1(X)$ under σ.

Proof. For any $\gamma \in \pi_1(X)$, the map $\bar{f}^*\gamma$ is a lifting of f, and so it must have the form $\bar{f}^*\gamma = (\phi_1(\gamma), \ldots, \phi_n(\gamma); \sigma_\gamma)\bar{f}^*$ for some functions σ and ϕ_i. Writing in coordinates gives (1). □

Theorem 2.5. *Let $f\colon X \to D_n(X)$ be an n-valued map and $\bar{f}^*$ its basic lifting. Let $\gamma, \delta \in \pi_1(X)$.*

If there exists $\mu \in \pi_1(X)$ with

$$\gamma\bar{f}_i^* = \mu^{-1}\delta\bar{f}_j^*\mu,$$

then $p\,\mathrm{Fix}(\gamma\bar{f}_i^*) = p\,\mathrm{Fix}(\delta\bar{f}_j^*)$.

If $p\,\mathrm{Fix}(\gamma\bar{f}_i^*)$ is nonempty, then the converse holds: if $p\,\mathrm{Fix}(\gamma\bar{f}_i^*) \cap p\,\mathrm{Fix}(\delta\bar{f}_j^*) \neq \emptyset$ then $\gamma\bar{f}_i^* = \mu^{-1}\delta\bar{f}_j^*\mu$ for some $\mu \in \pi_1(X)$.

Proof. Assume that $\gamma\bar{f}_i^* = \mu^{-1}\delta\bar{f}_j^*\mu$, and we will show that $p\,\mathrm{Fix}(\gamma\bar{f}_i^*) = p\,\mathrm{Fix}(\delta\bar{f}_j^*)$.

Let $x_0 \in p\,\mathrm{Fix}(\gamma\bar{f}_i^*)$. Then there exists $\tilde{x}_0 \in p^{-1}(x_0) \cap \mathrm{Fix}(\gamma\bar{f}_i^*)$ so

$$\tilde{x}_0 = \gamma\bar{f}_i^*(\tilde{x}_0) = \mu^{-1}\delta\bar{f}_j^*\mu(\tilde{x}_0).$$

Therefore $\mu\tilde{x}_0 = \delta\bar{f}_j^*\mu(\tilde{x}_0)$, that is $\mu\tilde{x}_0 \in \mathrm{Fix}(\delta\bar{f}_j^*)$, and thus $x_0 = p(\mu\tilde{x}_0) \in p\,\mathrm{Fix}(\delta\bar{f}_j^*)$. We have proved that $p\,\mathrm{Fix}(\gamma\bar{f}_i^*) \subseteq p\,\mathrm{Fix}(\delta\bar{f}_j^*)$. A symmetric argument establishes that $p\,\mathrm{Fix}(\gamma\bar{f}_i^*) = p\,\mathrm{Fix}(\delta\bar{f}_j^*)$.

Now, for the converse, we assume that $p\,\mathrm{Fix}(\gamma\bar{f}_i^*) \cap p\,\mathrm{Fix}(\delta\bar{f}_j^*)$ is nonempty and we will find the appropriate element μ.

Let $\tilde{x}_0 \in p^{-1}(x_0) \cap \mathrm{Fix}(\gamma\bar{f}_i^*)$ so $\gamma\bar{f}_i^*(\tilde{x}_0) = \tilde{x}_0$. Since $x_0 \in p\,\mathrm{Fix}(\delta\bar{f}_j^*)$, there exists $\mu\tilde{x}_0 \in p^{-1}(x_0) \cap \mathrm{Fix}(\delta\bar{f}_j^*)$ so that $\delta\bar{f}_j^*(\mu\tilde{x}_0) = \mu\tilde{x}_0$ and thus

$$\tilde{x}_0 = \gamma\bar{f}_i^*(\tilde{x}_0) = \mu^{-1}\delta\bar{f}_j^*\mu(\tilde{x}_0).$$

Since the liftings $\gamma\bar{f}_i^*$ and $\mu^{-1}\delta\phi_j(\mu)\bar{f}^*_{\sigma_\mu(j)}$ agree at a point and f is an n-valued map and therefore no two different liftings can take on the same value, they are the same lifting, that is, $i = \sigma_\mu(j)$ and

$$\gamma\bar{f}_i^* = \mu^{-1}\delta\phi_j(\mu)\bar{f}^*_{\sigma_\mu(j)} = \mu^{-1}\delta\bar{f}_j^*\mu$$

which completes the proof. □

Consequently, the fixed point classes are related to conjugacy of liftings in the following manner.

Corollary 2.6. *Let $(\alpha;\eta)\bar{f}^*, (\beta;\theta)\bar{f}^*\colon \tilde{X} \to F_n(\tilde{X},\pi)$ be liftings of $f\colon X \to D_n(X)$. If these liftings are conjugate, then*

$$p\,\mathrm{Fix}(\alpha_i \bar{f}^*_{\eta(i)}) = p\,\mathrm{Fix}(\beta_i \bar{f}^*_{\theta(i)})$$

for all $i = 1, \ldots n$. Conversely, if $p\,\mathrm{Fix}(\alpha_i \bar{f}^*_{\eta(i)}) \cap p\,\mathrm{Fix}(\beta_i \bar{f}^*_{\theta(i)}) \neq \emptyset$ for all $i = 1, \ldots n$, then $(\alpha;\eta)\bar{f}^*$ and $(\beta;\theta)\bar{f}^*$ are conjugate.

Theorem 2.7. *Given a lifting $(\alpha;\eta)\bar{f}^*$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$, the fixed point classes $p\,\mathrm{Fix}(\alpha_i \bar{f}^*_{\eta(i)})$ and $p\,\mathrm{Fix}(\alpha_j \bar{f}^*_{\eta(j)})$ for $i \neq j$ are disjoint.*

Proof. Let $x_0 \in p\,\mathrm{Fix}(\alpha_i \bar{f}^*_{\eta(i)}) \cap p\,\mathrm{Fix}(\alpha_j \bar{f}^*_{\eta(j)})$, then there exist $\tilde{x}_i, \tilde{x}_j \in p^{-1}(x_0)$ such that $\alpha_i \bar{f}^*_{\eta(i)}(\tilde{x}_i) = \tilde{x}_i$ and $\alpha_j \bar{f}^*_{\eta(j)}(\tilde{x}_j) = \tilde{x}_j$. But then $p\alpha_i \bar{f}^*_{\eta(i)}(\tilde{x}_i) = p\alpha_j \bar{f}^*_{\eta(j)}(\tilde{x}_j) = x_0$ and since $(\alpha;\eta)\bar{f}^*\colon \tilde{X} \to F_n(\tilde{X},\pi)$, then $i = j$. □

Thus each lifting class gives rise to n disjoint fixed point classes, some of which may be empty. However, fixed point classes arising from one lifting class may equal classes arising from a different lifting class, as we will demonstrate in Section 5.

Theorem 2.8. *If n-valued maps $f, g\colon X \to D_n(X)$ are homotopic, then $C(f) = C(g)$.*

Proof. Let $f, g\colon X \to D_n(X)$ be homotopic n-valued maps and let $H\colon X \times I \to D_n(X)$ be a homotopy such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. Let

$$(\alpha;\eta)\bar{f}^* = (\alpha_1 \bar{f}^*_{\eta(1)}, \ldots, \alpha_n \bar{f}^*_{\eta(n)})\colon \tilde{X} \to F_n(\tilde{X},\pi)$$

be a lifting of f. Lifting H to

$$\bar{H} = (\bar{h}_1, \ldots, \bar{h}_n)\colon \tilde{X} \times I \to F_n(\tilde{X},\pi)$$

such that $\bar{h}_i(\tilde{x},0) = \alpha_i \bar{f}^*_{\eta(i)}(\tilde{x})$ defines a lifting $\bar{g}$ of g by setting $\bar{H}(\tilde{x},1) = \bar{g}(\tilde{x})$. This establishes a one-to-one correspondence between the liftings of f and the liftings of g.

Suppose $(\beta;\theta)\bar{f}^*$ is a lifting that is conjugate to $(\alpha;\eta)\bar{f}^*$ via $\mu = (\mu_1, \ldots, \mu_n) \in \pi_1(X)^n$. Let $\bar{K} : \tilde{X} \times I \to F_n(\tilde{X},\pi)$ be the lifting of H such that $\bar{k}_i(\tilde{x},0) = \beta_i \bar{f}^*_{\theta(i)}$. Since $\mu_i^{-1}\bar{k}_i(\mu_i\tilde{x},0) = \bar{h}_i(\tilde{x},0)$ for all $\tilde{x} \in \tilde{X}$, we know that $\mu_i^{-1}\bar{k}_i(\mu_i\tilde{x},t) = \bar{h}_i(\tilde{x},t)$ for all $t \in [0,1]$ and thus, in particular, the liftings $\bar{h}_i(\tilde{x},1)$ and $\bar{k}_i(\tilde{x},1)$ of g are conjugate so the equivalence relation is preserved and therefore $C(f) = C(g)$. □

3 The Construction of Dugardein

We continue to denote the universal covering space of X by $p\colon \tilde{X} \to X$. Given an n-valued map $f\colon X \to D_n(X)$, Gert-Jan Dugardein defined a map $\hat{f}\colon \tilde{X} \to F_n(\tilde{X}, \pi)$ as follows.[2] The map $fp\colon \tilde{X} \to D_n(X)$ lifts to $F_n(X)$ and therefore it splits as $(f_1, \ldots, f_n)\colon \tilde{X} \to F_n(X)$ where $f_i\colon \tilde{X} \to X$ for each i. The ordering can be chosen so that each f_i lifts to $\tilde{f}_i\colon \tilde{X} \to \tilde{X}$ such that $\tilde{f}_i(\tilde{x}^*) = \tilde{x}_i^{(0)}$, the basepoints of Section 2. Therefore, $(f_1, \ldots, f_n)$ lifts to $\hat{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)\colon \tilde{X} \to F_n(\tilde{X})$ such that $\hat{f}(\tilde{x}^*) = \tilde{x}^{(0)}$. Since $p^n(\tilde{f}_1, \ldots, \tilde{f}_n) \in D_n(X)$, then $p\tilde{f}_i \neq p\tilde{f}_j$ for $i \neq j$ and therefore we may consider $\hat{f}$ as a map $\hat{f} : \tilde{X} \to F_n(\tilde{X}, \pi)$.

It is clear from the definition that $\hat{f} : \tilde{X} \to F_n(\tilde{X}, \pi)$ is a lifting of $f : X \to D_n(X)$. The basic lifting $\bar{f}^*$ is also such a lifting and $\hat{f}(\tilde{x}^*) = \bar{f}^*(\tilde{x}^*) = \tilde{x}^{(0)}$, so they are the same map. Moreover, since $(\tilde{f}_1, \ldots, \tilde{f}_n)$ and $(\bar{f}_1, \ldots, \bar{f}_n)$ are two splittings of $\bar{f} = \hat{f}$ that correspond at the basepoints, then $\tilde{f}_i = \bar{f}_i$ for $i = 1, \ldots, n$. Consequently, the results of this section, that concern the subsets $p\,\mathrm{Fix}(\alpha_i \tilde{f}_i)$ of $\mathrm{Fix}(f)$, apply as well to the fixed point classes that were defined in Section 2 as the sets $p\,\mathrm{Fix}(\alpha_i \bar{f}_i^*)$.

For a map $f\colon X \to D_n(X)$, employing the definition of Schirmer in [14], we will say that $x_0, x_1 \in \mathrm{Fix}(f)$ are *S-equivalent* if there is a map $c\colon I = [0,1] \to X$ from x_0 to x_1 such that, for the splitting $fc = \{c_1, \ldots, c_n\}\colon I \to D_n(X)$, some c_k is a path from x_0 to x_1 and c_k is homotopic to c relative to the endpoints.

Theorem 3.1 (Dugardein)**.** *Fixed points x_0, x_1 of $f\colon X \to D_n(X)$ are S-equivalent if and only if there exists $i \in \{1, \ldots, n\}$ and $\alpha_i \in \pi_1(X)$ such that $x_0, x_1 \in p\,\mathrm{Fix}(\alpha_i \tilde{f}_i)$.*

Proof. Suppose $x_0, x_1 \in p\,\mathrm{Fix}(\alpha_i \tilde{f}_i)$ so there exist $\tilde{x}_0 \in p^{-1}(x_0), \tilde{x}_1 \in p^{-1}(x_1)$ in $\tilde{X}$ such that $\tilde{x}_0 = \alpha_i \tilde{f}_i(\tilde{x}_0)$ and $\tilde{x}_1 = \alpha_i \tilde{f}_i(\tilde{x}_1)$. Let $\tilde{c}\colon I \to \tilde{X}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$, then $\tilde{c}$ is homotopic to $\alpha_i \tilde{f}_i \tilde{c}$ relative to the endpoints by a homotopy $\tilde{H}\colon I \times I \to \tilde{X}$. Let $p(\tilde{c}) = c$, then $fc\colon I \to D_n$ splits as $fc = \{c_1, \ldots, c_n\}$. We have a commutative diagram:

[2] The construction and its properties were presented at the conference "Nielsen Theory and Related Topics" held in Rio Claro, Brazil in July, 2016. Dugardein has made the slides of his talk available to the authors and given them permission to include this material in the present paper.

$$\begin{tikzcd}
 & \tilde{X} \arrow[r, "(f_1,\ldots,f_n)"] \arrow[d, "p"] & F_n(X) \arrow[d, "q"] \\
I \arrow[ur, "\tilde{c}"] \arrow[r, "c"] & X \arrow[r, "f"] & D_n(X)
\end{tikzcd}$$

Now $p\tilde{H}\colon I \times I \to X$ is a homotopy between c and $p\alpha_i\tilde{f}_i\tilde{c}$. On the other hand,

$$p\alpha_i\tilde{f}_i\tilde{c} = p\tilde{f}_i\tilde{c} = f_i\tilde{c} \in q(f_1, \ldots, f_n)\tilde{c} = fc$$

so $p\alpha_i\tilde{f}_i\tilde{c} = c_k$ for some $k \in \{1, \ldots, n\}$ and therefore x_0 and x_1 are S-equivalent.

If fixed points x_0, x_1 of $f\colon X \to D_n(X)$ are S-equivalent, then there is a map $c\colon I \to X$ from x_0 to x_1 such that, for the splitting $fc = \{c_1, \ldots, c_n\}\colon I \to D_n(X)$, some c_k is a path from x_0 to x_1 and c_k is homotopic to c relative to the endpoints. Let $H\colon I \times I \to X$ be a homotopy from c_k to c relative to the endpoints, that is, $H(0,t) = x_0, H(1,t) = x_1$ for all $t \in I$ and $H(s,0) = c_k(s), H(s,1) = c(s)$ for all $s \in I$. Choose some $\tilde{x}_0 \in p^{-1}(x_0)$. Since $(\tilde{f}_1, \ldots, \tilde{f}_n)$ is a lifting of f and x_0 is a fixed point of f, then there exists $i \in \{1, \ldots, n\}$ such that $\tilde{f}_i(\tilde{x}_0) \in p^{-1}(x_0)$ and therefore $\alpha_i \in \pi_1(X)$ such that $\alpha_i\tilde{f}_i(\tilde{x}_0) = \tilde{x}_0$. Let $\tilde{H}\colon I \times I \to \tilde{X}$ be the lifting of H to $\tilde{x}_0$ such that $\tilde{H}(0,t) = \tilde{x}_0$ for all $t \in I$. Define $\tilde{x}_1 = \tilde{H}(1,0)$ and thus $\tilde{H}(1,t) = \tilde{x}_1$ for all $t \in I$. The restriction of $\tilde{H}$ to $I \times \{1\}$ lifts the path c to $\tilde{x}_0 = \alpha_i\tilde{f}_i(\tilde{x}_0)$ and so, in particular, $\tilde{x}_1 = \alpha_i\tilde{f}_i(\tilde{x}_1)$. We have proved that $x_0, x_1 \in p\,\mathrm{Fix}(\alpha_i\tilde{f}_i)$. $\square$

Thus by Theorem 3.1 the fixed point classes $p\,\mathrm{Fix}(\alpha_i\bar{f}_i^*)$ defined in Section 2 are the same subsets of $\mathrm{Fix}(f)$ as those of Schirmer in [14]. Schirmer in [14] defined the *Nielsen number* $N(f)$ of an n-valued map to be the number of fixed point classes of non-zero index. By Theorem 2.7 there are at most n fixed point classes corresponding to each equivalence class of liftings, and so we conclude that

Proposition 3.2. *For any n-valued map $f\colon X \to D_n(X)$, we have $N(f) \leq n \cdot C(f)$.*

4 Reidemeister Equivalence

Recall the definition of conjugacy of two liftings $(\alpha;\eta)\bar{f}^*$ and $(\beta;\theta)\bar{f}^*$: for each $i = 1, \ldots, n$ there exists $\mu_i \in \pi_1(X)$ such that

$$\alpha_i\bar{f}^*_{\eta(i)} = \mu_i^{-1}\beta_i\bar{f}^*_{\theta(i)}\mu_i.$$

Given an n-valued map $f\colon X \to D_n(X)$ and the corresponding functions $\phi_i\colon \pi_1(X) \to \pi_1(X)$ and $\sigma\colon \pi_1(X) \to \Sigma_n$ from Lemma 2.4 we have

$$\bar{f}^*_{\theta(i)}\mu_i = \phi_{\theta(i)}(\mu_i)\bar{f}^*_{\sigma_{\mu_i}(\theta(i))}$$

so $(\alpha;\eta)\bar{f}^*$ and $(\beta;\theta)\bar{f}^*$ are conjugate if and only if

$$\alpha_i\bar{f}^*_{\eta(i)} = \mu_i^{-1}\beta_i\phi_{\theta(i)}(\mu_i)\bar{f}^*_{\sigma_{\mu_i}(\theta(i))}$$

for all $i = 1, \ldots, n$. Thus it must be that

$$\alpha_i = \mu_i^{-1}\beta_i\phi_{\theta(i)}(\mu_i) \qquad \text{and} \qquad \eta(i) = \sigma_{\mu_i}(\theta(i))$$

for all $i = 1, \ldots, n$.

For an n-valued map $f\colon X \to D_n(X)$, define $(\alpha, i), (\beta, j) \in \pi_1(X) \times \{1, \ldots, n\}$ to be *f-Reidemeister equivalent*, written $[\alpha]_i = [\beta]_j$ if there exists $\mu \in \pi_1(X)$ such that

$$\alpha = \mu^{-1}\beta\phi_j(\mu) \qquad \text{and} \qquad i = \sigma_\mu(j)$$

The definition immediately gives:

Theorem 4.1. *Liftings $(\alpha;\eta)\bar{f}^*$ and $(\beta;\theta)\bar{f}^*$ are conjugate if and only if $[\alpha_i]_{\eta(i)} = [\beta_i]_{\theta(i)}$ for all $i = 1, \ldots, n$.*

The function σ is a homomorphism, and although the individual ϕ_i are not necessarily homomorphisms (see Section 5), there is still some regularity.

Lemma 4.2. *Let I denote the trivial permutation. With the notation as in Lemma 2.4, we have:*

$$\sigma_{\alpha\beta} = \sigma_\beta \circ \sigma_\alpha \tag{2}$$
$$\phi_i(\alpha\beta) = \phi_i(\alpha)\phi_{\sigma_\alpha(i)}(\beta) \tag{3}$$
$$\sigma_1 = I \tag{4}$$
$$\phi_i(1) = 1 \tag{5}$$
$$\sigma_{\gamma^{-1}} = \sigma_\gamma^{-1} \tag{6}$$
$$\phi_i(\gamma^{-1}) = \phi_{\sigma_\gamma^{-1}(i)}(\gamma)^{-1} \tag{7}$$

Proof. Equations (2) and (3) are consequences of the definition of the ϕ_i, as follows: From the equation $\bar{f}^*_i(\alpha\tilde{x}) = \phi_i(\alpha)\bar{f}^*_{\sigma_\alpha(i)}(\tilde{x})$ we have $\bar{f}^*_i(\alpha\beta\tilde{x}) = \phi_i(\alpha\beta)\bar{f}^*_{\sigma_{\alpha\beta}(i)}(\tilde{x})$ but also

$$\bar{f}^*_i(\alpha\beta\tilde{x}) = \phi_i(\alpha)\bar{f}^*_{\sigma_\alpha(i)}(\beta\tilde{x}) = \phi_i(\alpha)\phi_{\sigma_\alpha(i)}(\beta)\bar{f}^*_{\sigma_{\sigma_\beta(\sigma_\alpha(i))}}(\tilde{x})$$

from which it follows that $\phi_i(\alpha\beta)\bar{f}^*_{\sigma_{\alpha\beta}(i)} = \phi_i(\alpha)\phi_{\sigma_\alpha(i)}(\beta)\bar{f}^*_{\sigma_{\sigma_\beta(\sigma_\alpha(i))}}$ for all i and thus (2) and (3) hold. Since

$$\bar{f}^*_i(1\tilde{x}) = \bar{f}^*_i(\tilde{x}) = 1\bar{f}^*_i(\tilde{x}) = \phi_i(1)\bar{f}^*_i(\tilde{x})$$

for all i and $\tilde{x}$, then (4) and (5) hold. From (2) and (4) we have

$$I = \sigma_1 = \sigma_{\gamma\gamma^{-1}} = \sigma_{\gamma^{-1}} \circ \sigma_\gamma$$

which proves (6). From (5), (3) and (6) we get

$$1 = \phi_i(1) = \phi_i(\gamma^{-1}\gamma) = \phi_i(\gamma^{-1})\phi_{\sigma_{\gamma^{-1}}(i)}(\gamma) = \phi_i(\gamma^{-1})\phi_{\sigma_\gamma^{-1}(i)}(\gamma)$$

which gives (8). □

Proposition 4.3. *The f-Reidemeister relation on $\pi_1(X)\times\{1,\ldots,n\}$ is an equivalence relation.*

Proof. For reflexivity, we must show that $[\alpha]_i = [\alpha]_i$. That is, we must find γ with $\sigma_\gamma(i) = i$ and $\alpha = \gamma^{-1}\alpha\phi_i(\gamma)$. Letting $\gamma = 1$, by (4) we have $\sigma_1 = I$ and so $\sigma_1(i) = i$, and by (5) we have $\phi_i(1) = 1$. Thus

$$\gamma^{-1}\alpha\phi_i(\gamma) = 1\alpha\phi_i(1) = \alpha.$$

For symmetry, assume that there is some γ with $\sigma_\gamma(j) = i$ and $\alpha = \gamma^{-1}\beta\phi_j(\gamma)$. We claim that γ^{-1} realizes the symmetric equivalence. That is, we must show that $\sigma_{\gamma^{-1}}(i) = j$ and $\beta = \gamma\alpha\phi_i(\gamma^{-1})$. By (6) we have $\sigma_{\gamma^{-1}}(i) = \sigma_\gamma^{-1}(i) = j$ as desired. Also we have:

$$\beta = \gamma\alpha\phi_j(\gamma)^{-1} = \gamma\alpha\phi_{\sigma_{\gamma^{-1}}(i)}(\gamma)^{-1} = \gamma\alpha\phi_i(\gamma^{-1})$$

by (7), and symmetry is proved. For transitivity, assume that $[\alpha]_i = [\beta]_j$ and $[\beta]_j = [\delta]_k$. Then there is some γ with $\sigma_\gamma(j) = i$ and $\alpha = \gamma^{-1}\beta\phi_j(\gamma)$ and some λ with $\sigma_\lambda(k) = j$ and $\beta = \lambda^{-1}\delta\phi_k(\lambda)$. We claim that $\lambda\gamma$ realizes the equivalence of $[\alpha]_i$ and $[\delta]_k$. That is, we will show that $\sigma_{\lambda\gamma}(k) = i$ and $\alpha = (\lambda\gamma)^{-1}\delta\phi_k(\lambda\gamma)$. By (2) we have

$$\sigma_{\lambda\gamma}(k) = \sigma_\gamma(\sigma_\lambda(k)) = \sigma_\gamma(j) = i$$

as and by (3) we have

$$(\lambda\gamma)^{-1}\delta\phi_k(\lambda\gamma) = \gamma^{-1}\lambda^{-1}\delta\phi_k(\lambda)\phi_{\sigma_\lambda(k)}(\gamma) = \gamma^{-1}\beta\phi_j(\gamma) = \alpha. \quad \square$$

We define the *Reidemeister number* $R(f)$ of $f\colon X \to D_n(X)$ to be the number of equivalence classes of $\pi_1(X) \times \{1,\ldots,n\}$ under the f-Reidemeister relation.

The Reidemeister number and lifting class number are related as follows.

Theorem 4.4. *Let $f : X \to D_n(X)$ be an n-valued map. Then $C(f) = R(f)^n$.*

Proof. We prove the theorem in two steps: first we show that $C(f) \leq R(f)^n$, and then that $C(f) \geq R(f)^n$.

Let $\mathcal{C}(f)$ denote the set of lifting classes of f, and $\mathcal{R}(f)$ the set of Reidemeister classes. Define $b : \mathcal{C}(f) \to \mathcal{R}(f)^n$ by $b([\alpha_1, \dots, \alpha_n; \eta]) = ([\alpha_1]_{\eta(1)}, \dots, [\alpha_n]_{\eta(n)})$. By Theorem 4.1, this function is well-defined on lifting classes, and also injective. Since there is an injection $\mathcal{C}(f) \to \mathcal{R}(f)^n$, we have $C(f) \leq R(f)^n$.

Now we show that $C(f) \geq R(f)^n$. Let A be a set of representatives for the Reidemeister classes, so $\mathcal{R}(f) = A = \{\alpha_1, \alpha_2, \dots\}$ and integers $\{k_1, k_2, \dots\}$ so that $\mathcal{R}(f)$ is the following set of distinct elements:

$$\mathcal{R}(f) = \{[\alpha_1]_{k_1}, [\alpha_2]_{k_2}, \dots\}. \tag{8}$$

By our construction, A and $\{k_1, k_2, \dots\}$ each have cardinality equal to $R(f)$, which may or may not be finite. Since there are no repetitions in this list of elements of $\mathcal{R}(f)$, any equality $[\alpha_j]_i = [\alpha_l]_i$ will imply that $j = l$.

Let $S \subset \pi_1(X)^n \rtimes \Sigma_n$ be the set of elements $(\alpha_{j_1}, \dots, \alpha_{j_n}; 1)$ where 1 denotes the trivial permutation, and each $j_i \in \{1, \dots, R(f)\}$. (When $R(f)$ is infinite, each $j_i \in \{1, 2, \dots\}$.) We will demonstrate that the elements of S each represent different lifting classes:

Let $(\alpha_{j_1}, \dots, \alpha_{j_n}; 1), (\alpha_{l_1}, \dots, \alpha_{l_n}; 1) \in S$ with $[\alpha_{j_1}, \dots, \alpha_{j_n}; 1] = [\alpha_{l_1}, \dots, \alpha_{l_n}; 1]$. Then we will show that $\alpha_{j_i} = \alpha_{l_i}$ for each i. By Theorem 4.1, we have $[\alpha_{j_i}]_i = [\alpha_{l_i}]_i$ for each $i = 1, \dots, n$. Since the listing of elements in (8) has no repetitions, this means that $j_i = l_i$ for each i, and thus that $\alpha_{j_i} = \alpha_{l_i}$ for each i.

We have shown that each element of S represents a different lifting class. The set S is clearly in bijective correspondence with A^n, and since each element of S represents a different lifting class, there are at least $(\#A)^n = R(f)^n$ lifting classes. Thus if $R(f)$ is infinite, then there are infinitely many lifting classes, and if $R(f)$ is finite, then $C(f) \geq R(f)^n$. □

Then Proposition 3.2 implies:

Corollary 4.5. *For any n-valued map $f\colon X \to D_n(X)$, we have $N(f) \leq n \cdot R(f)^n$.*

Since $C(f) = R(f)^n$ and $R(f)$ is a nonnegative integer, Theorem 2.8 implies:

Corollary 4.6. *If n-valued maps $f, g\colon X \to D_n(X)$ are homotopic, then $R(f) = R(g)$.*

5 The Circle

Represent S^1 as the complex numbers of norm one and define $f\colon S^1 \to D_2(S^1)$ by letting $f(z)$ be the two square roots of z. Then the map $(\bar{f}_1, \bar{f}_2)\colon \mathbb{R} \to F_2(\mathbb{R}, \mathbb{Z})$ defined by $\bar{f}_1(t) = \frac{t}{2}$ and $\bar{f}_2(t) = \frac{t+1}{2} = \bar{f}_1(t) + \frac{1}{2}$ is a lifting of f. Since the covering transformations are $k \in \mathbb{Z}$ acting on $\mathbb{R}$ by $k(t) = t + k$, then

$$\bar{f}_i(kt) = \bar{f}_i(k(t)) = \bar{f}_i(t+k) = \bar{f}_i(t) + \frac{k}{2}$$

for $i = 1, 2$.

Thus if k is even, then $\bar{f}_i(kt) = \frac{k}{2}\,\bar{f}_i(t)$. If k is odd, then

$$\bar{f}_1(kt) = \bar{f}_1(t) + \frac{k}{2} = \bar{f}_1(t) + \frac{1}{2} + \frac{k-1}{2} = \bar{f}_2(t) + \frac{k-1}{2}$$

and

$$\bar{f}_2(kt) = \bar{f}_2(t) + \frac{k}{2} = \bar{f}_1(t) - \frac{1}{2} + \frac{k+1}{2} = \bar{f}_1(t) + \frac{k+1}{2}$$

and we conclude that $\bar{f}_1(kt) = \frac{k-1}{2}\,\bar{f}_2(t)$ and $\bar{f}_2(kt) = \frac{k+1}{2}\,\bar{f}_1(t)$. Therefore, σ_k is the identity permutation of $\{1, 2\}$ if k is even and the other permutation if k is odd. The functions $\phi_i\colon \mathbb{Z} \to \mathbb{Z}$ are defined by $\phi_1(k) = \frac{k}{2}$ if k is even and $\phi_1(k) = \frac{k-1}{2}$ if k is odd and $\phi_2(k) = \frac{k}{2}$ if k is even and $\phi_2(k) = \frac{k+1}{2}$ if k is odd. We note that the ϕ_i are not homomorphisms; for instance $\phi_1(1) = 0$ but $\phi_1(1+1) = 1$.

We will show that $R(f) = 1$. That is, for $\alpha, \beta \in \mathbb{Z}$ and $i, j \in \{1, 2\}$, we always have $[\alpha]_i = [\beta]_j$. (This provides an example in which fixed point classes arising from different lifting classes can be equal, as mentioned after Theorem 2.7.)

We must find some $k \in \mathbb{Z}$ with $\sigma_k(j) = i$ and $\alpha = -k + \beta + \phi_j(k)$. We treat the cases $i = j$ and $i \neq j$ separately. If $i = j$ we need $\sigma_k(i) = i$ and so we will choose k to be even. Thus both $\phi_j(k) = k/2$ and so we must choose k so that $\alpha = -k + \beta + k/2$, that is, $k = 2(\beta - \alpha)$. We verify that

$$-k + \beta + k/2 = -2(\beta - \alpha) + \beta + \beta - \alpha = \alpha.$$

If $i \neq j$ the cases $j = 1$ and $j = 2$ are slightly different. If $j = 1$ let $k = 2(\beta - \alpha) - 1$. Then k is odd so $\sigma_k(j) = i$ and $\phi_j(k) = \frac{k-1}{2}$ and we have:

$$-k + \beta + \frac{k-1}{2} = -2(\beta - \alpha) - 1 + \beta + \beta - \alpha = \alpha.$$

When $j = 2$ we choose $k = 2(\beta - \alpha) + 1$. Thus in all cases we have $[\alpha]_i = [\beta]_j$, and so $R(f) = 1$.

More generally, define an n-valued map $f\colon S^1 \to D_n(S^1)$ by letting $f(z)$ be the set of n-th roots of z^d for an integer $d \neq n$. Thus $f = \phi_{n,d}$ in the notation of [2]. We will prove that $[\alpha]_i = [\beta]_j$ if and only if $\alpha + i = \beta + j \mod (d-n)$ and therefore that $R(f) = |d-n|$.

Define liftings $\bar{f}_1^*, \ldots, \bar{f}_n^*$ by setting

$$\bar{f}_j^*(t) = \frac{dt + j - 1}{n}.$$

To compute $\bar{f}_j^*(t+k)$, divide dk by n to obtain integers q, r with $dk = qn + r$. Then we have:

$$\begin{aligned}\bar{f}_j^*(t+k) &= \frac{dt + qn + r + j - 1}{n} = \frac{dt + r + j - 1}{n} + q \\ &= \begin{cases} \bar{f}_{j+r}^*(t) + q & \text{if } j + r \leq n \\ \bar{f}_{j+r-n}^*(t) + q + 1 & \text{if } j + r > n \end{cases}\end{aligned}$$

Thus $\sigma_k(j)$ is either $j + r$ or $j + r - n$. In particular, if $\sigma_k(j) \geq j$, then $\sigma_k(j) = j + r$. Of the two cases in the formula above, we will only need the case where $j + r \leq n$. In this case we have

$$\sigma_k(j) = j + r \qquad \phi_j(k) = q.$$

Therefore if $j + r \leq n$ we compute

$$d(\phi_j(k) - k) = d\phi_j(k) - dk = dq - (qn + r) = (d-n)q - r.$$

Now let $\alpha, \beta \in \mathbb{Z}$ and $i, j \in \{1, \ldots .n\}$. We will prove that $[\alpha]_i = [\beta]_j$ if and only if $\alpha + i = \beta + j \mod d - n$. Assume that $[\alpha]_i = [\beta]_j$. Since the Reidemeister relation is symmetric, we may assume that $i \geq j$. Then there is some $k \in \mathbb{Z}$ with $\sigma_k(j) = i$ and $\alpha = -k + \beta + \phi_j(k)$. As above, divide dk by n to obtain $dk = qn + r$. Since $i = \sigma_k(j) \geq j$, we have $i = j + r$. Since $j + r = i \leq n$ then

$$d(\alpha - \beta) = d(\phi_j(k) - k) = (d-n)q - r = (d-n)q - (i-j).$$

and thus $d\alpha + i = d\beta + j \mod d - n$.

For the converse, assume that $d\alpha + i = d\beta + j \mod d - n$ where $\alpha, \beta \in \mathbb{Z}$ and $i, j \in \{1, \ldots, n\}$ with $i \geq j$. Then there is some q such that $d(\alpha - \beta) = (d-n)q + (j-i)$. Let $r = i - j$, and we have $d\alpha - d\beta = dq - nq - r$ so $nq + r$ is a multiple of d and therefore there exists $k \in \mathbb{Z}$ such that $dk = nq + r$. We have $j + r = i \leq n$ so

$$d(\alpha - \beta) = (d-n)q + r = d(\phi_j(k) - k)$$

which means that $\alpha - \beta = \phi_j(k) - k$ so $[\alpha]_i = [\beta]_j$.

We have proved that $[\alpha]_i = [\beta]_j$ if and only if $d\alpha + i = d\beta + j \mod d - n$ so there are $|d - n|$ Reidemeister classes characterized by the remainder of $d\alpha + i \mod d - n$ and thus $R(f) = |d - n|$. The Reidemeister number is a homotopy invariant by Theorem 4.6. By Theorem 3.1 of [2], every n-valued self-map $f : S^1 \to D_n(S^1)$ is homotopic to $\phi_{n,d}$ for some integer d, and this d is called the *degree* of f. We have shown:

Theorem 5.1. *Let $f : S^1 \to D_n(S^1)$ be an n-valued map of degree d. Then $R(f) = |d - n|$.*

By Theorem 4.4 we have:

Corollary 5.2. *Let $f : S^1 \to D_n(S^1)$ be an n-valued map of degree d. Then $C(f) = |d - n|^n$.*

Recall from Section 2 that liftings $(\alpha; \eta)\bar{f}^*$ and $(\beta; \theta)\bar{f}^*$ are said to be uniformly conjugate via $\gamma \in \pi_1(X)$ if

$$\alpha_i \bar{f}^*_{\eta(i)} = \gamma^{-1} \beta_i \bar{f}^*_{\theta(i)} \gamma$$

for each $i = 1, \ldots, n$ and that the number of uniform conjugacy classes is denoted by $C_u(f)$.

Theorem 5.3. *Let $f : S^1 \to D_n(S^1)$ be any map on the circle. Then $C_u(f) = \infty$.*

Proof. Let $\kappa_1, \kappa_2 \in \pi_1(S^1)$ as covering transformations so $\kappa_i(t) = t + k_i$ with $k_1 \neq k_2$. We will show that liftings $(\kappa_1, 1 \ldots, 1; 1)\bar{f}^*$ and $(\kappa_2, 1 \ldots, 1; 1)\bar{f}^*$ are not uniformly conjugate and therefore there are an infinite number of equivalence classes with respect to uniform conjugacy. To prove this, suppose the liftings were uniformly conjugate, then there exists $\mu \in \pi_1(S^1)$ with $\mu(t) = t + m$ such that

$$\kappa_1 \bar{f}^*_1 = \mu^{-1} \kappa_2 \bar{f}^*_1 \mu$$

and

$$\bar{f}^*_2 = \mu^{-1} \bar{f}^*_2 \mu.$$

From the second equation we have

$$\frac{dt + 1}{n} = \frac{d(t + m) + 1}{n} - m$$

which implies that $m = 0$ which in the first equation would require that $k_1 = k_2$.

The proof of Theorem 2.8 implies that the number of equivalence classes of liftings with regard to uniform conjugacy is a homotopy invariant, so every n-valued self-map of S^1 has an infinite number of such classes. $\square$

More generally, if $f\colon X \to D_n(X)$ where $\pi_1(X)$ is infinite and some $\bar{f}_i^* = \mu^{-1}\bar{f}_i^*\mu$ implies $\mu = 1$, then f has an infinite number of equivalence classes with regard to uniform conjugacy, that is, $C_u(f) = \infty$.

6 The Orbit Configuration Space and the Universal Cover

The single-valued theory of lifting classes, Reidemeister classes, and twisted conjugacy is defined in terms of the universal cover $\tilde{X}$ of X and the induced homomorphism $f_\# : \pi_1(X) \to \pi_1(X)$. The most direct generalization to the n-valued setting would seem to involve the universal cover of $D_n(X)$ and the induced homomorphism $f_\# : \pi_1(X) \to \pi_1(D_n(X))$. In this section we discuss these ideas and describe why we have instead opted to use the orbit configuration space. We will show that for manifolds of high dimensions (at least 3), the two approaches are the same. In dimensions 1 and 2 (as long as X is not the circle), the orbit configuration space approach is simpler than the universal cover, but still includes all data necessary to compute the Nielsen theory of an n-valued map.

First we briefly review the Galois Correspondence for covering spaces (see [10, Theorem 1.38]) which states that there is a bijective correspondence between isomorphism classes of connected covering spaces over X and conjugacy classes of subgroups of $\pi_1(X)$. We summarize the specific facts that we will need in a lemma:

Lemma 6.1. *Let A and X be connected, locally connected, and semilocally simply connected. Let $u : \tilde{A} \to A$ be the universal cover of A, and let $p : B \to A$ be some other connected cover. Then:*

a. *There is a covering map $r : \tilde{A} \to B$ with $u = p \circ r$, and the covering group of $p : B \to A$ is $\pi_1(A)/N$ for some normal subgroup $N \leq \pi_1(A)$. If N is trivial, then r is a homeomorphism.*

b. *If $f : X \to A$ is a map and $\tilde{f} : \tilde{X} \to \tilde{A}$ is a lifting of f to universal covers, then there is a lifting $\bar{f} : \tilde{X} \to B$ with $\bar{f} = r \circ \tilde{f}$.*

c. *If $f_\# : \pi_1(X) \to \pi_1(A)$ is the induced homomorphism on the fundamental group, then there is an induced homomorphism $\phi : \pi_1(X) \to \pi_1(A)/N$ with $\phi = q \circ f_\#$, where q is the canonical surjection $q : \pi_1(X) \to \pi_1(X)/N$.*

We will apply the lemma above to the setting of maps $f : X \to D_n(X)$ and the covers $p^n : F_n(\tilde{X}, \pi) \to D_n(X)$ and the universal cover $q : \tilde{D}_n(X) \to D_n(X)$.

As a preliminary we must establish that $p^n : F_n(\tilde{X}, \pi) \to D_n(X)$ is a connected cover, and that $D_n(X)$ has the appropriate connectedness properties for covering space theory. We begin with connectedness of $F_n(\tilde{X}, \pi)$. Our argument closely resembles a similar argument for $F_n(X)$, which we present as a warm-up. The idea is due to Farber, in [6, Section 8].

Farber's result is important in the topological theory of robot motion planning. The following result, when X is a 1-complex other than the circle or interval, essentially says that any labeled set of n robots moving along a track with junctions can be rearranged without colliding to move to any desired locations.

Lemma 6.2. *Let X be a connected polyhedron not homeomorphic to the interval or circle. Then $F_n(X)$ is path connected.*

Proof. The fact that X is not the interval or circle means that some subdivision of X must have a vertex which meets at least 3 edges. Such a vertex is called an *essential* vertex in [6]. Let v be an essential vertex of X, let e be one of the edges meeting v, and let $z = (z_1, \ldots, z_n) \in F_n(X)$ be some ordered n-configuration with $z_i \in e$ for each i, with the points ordered by increasing distance from v.

To show that $F_n(X)$ is connected, it suffices to show that any other ordered n-configuration $x = (x_1, \ldots, x_n) \in F_n(X)$ can be connected to z by a path in $F_n(X)$. This path in $F_n(X)$ would consist of n paths γ_i connecting x_i to z_i such that $\gamma_i(t) \neq \gamma_j(t)$ for any t and all $i \neq j$. We imagine such a set of paths as representing a continuous motion of the points x_i into z_i where the n points never collide.

It is clear that the points $\{x_1, \ldots, x_n\}$, if we disregard their ordering, can be moved without colliding into the points $\{z_1, \ldots, z_n\}$. To achieve this choose some metric on X and order the x_i by increasing distance from v. Then we may move the points without colliding into the edge e one at a time starting with those nearest to v. (Note this argument shows that $D_n(X)$ is connected for any connected polyhedron, even the circle or interval.)

To show that $F_n(X)$ is connected, it remains only to show that the configuration $(z_1, \ldots, z_n)$ can be moved without collision into any permutation of itself. This is accomplished by Farber's algorithm described in detail in [6] (a similar procedure is used in [15]). Briefly, the points may rearrange without colliding by using the essential

vertex as a three-way road junction: Let e_2 and e_3 be two other edges meeting v. If for example z_3 and z_4 wish to exchange their positions, then first z_1 and z_2 can move into e_2. Then z_3 moves into e_3, then z_4 moves into e_2, then z_3 moves back into e followed by z_4 and finally by z_2 and z_1. In this way any desired permutation of the z_i can be achieved by noncolliding paths. □

In the following result we generalize Farber's procedure to the setting of the orbit configuration space. There is a motion-planning interpretation of this theorem: if we have n objects moving along tracks which are arranged like a covering space, for example vertically stacked tracks as in a parking garage, then any labeled set of n robots moving along these tracks can be rearranged into any desired locations without ever colliding or moving directly above or below each other.

Theorem 6.3. *Let X be a connected polyhedron not homeomorphic to the interval or circle. Then $F_n(\tilde{X}, \pi)$ is path connected.*

Proof. We will mimic the argument used in Lemma 6.2. Let v be an essential vertex X with some incident edge e, and choose $\tilde{v} \in \tilde{X}$ with $p(\tilde{v}) = v$ and the edge $\tilde{e}$ incident at $\tilde{v}$ with $p(\tilde{e}) = e$. Let $(\tilde{z}_1, \ldots, \tilde{z}_n) \in F_n(X, \pi)$ with $\tilde{z}_i \in \tilde{e}$ for each i, and we will show that any configuration $(\tilde{x}_1, \ldots, \tilde{x}_n) \in F_n(X, \pi)$ can be connected to $(\tilde{z}_1, \ldots, \tilde{z}_n)$ by a path in $F_n(X, \pi)$.

We imagine this path in $F_n(X, \pi)$ as representing a continuous motion of the points $\tilde{x}_i$ into the points $\tilde{z}_i$ such that the projections $x_i = p(\tilde{x}_i)$ never collide in X. By first mimicking exactly the argument from Lemma 6.2, we can move each of the points $\tilde{x}_i$ into some covering translations of $\tilde{e}$, reaching points $\gamma_i \tilde{z}_i$ for some $\gamma_i \in \pi_1(X)$ such that the projections are noncolliding during the motion.

It remains to show that the configuration $(\gamma_1 \tilde{z}_1, \ldots, \gamma_n \tilde{z}_n)$ can be moved to $(\tilde{z}_1, \ldots, \tilde{z}_n)$ with noncolliding projections. Since we may achieve any permutation of the $\gamma_i z_i$ by moving points so that their projections move according to the three-way junction at v, it will be enough to show that $(\gamma_1 \tilde{z}_1, \gamma_2 \tilde{z}_2, \ldots, \gamma_n \tilde{z}_n)$ can be moved to $(\tilde{z}_1, \gamma_2 \tilde{z}_2, \ldots, \gamma_n \tilde{z}_n)$ with noncolliding projections.

Viewing $\gamma_1 \in \pi_1(X)$ as a loop provides a path from $\gamma_1 \tilde{z}_1$ to $\tilde{z}_1$. Again using the three-way junction at v we can move $\gamma_1 \tilde{z}$ along this path to $\tilde{z}_1$ so that no collisions occur in the projection. (If a collision of projections is about to occur in e, the two points can use the three-way junction to exchange positions before continuing.) Thus $(\gamma_1 \tilde{z}_1, \gamma_2 \tilde{z}_2, \ldots, \gamma_n \tilde{z}_n)$ can be moved to $(\tilde{z}_1, \gamma_2 \tilde{z}_2, \ldots, \gamma_n \tilde{z}_n)$ with noncolliding projections as desired. □

In the case where X is the interval or circle, $F_n(X, \pi)$ is indeed disconnected.

Example 6.4. Let X be the interval $[0, 1]$. Then $\tilde{X} = X$ and $\pi_1(X)$ is trivial, so $F_n(X, \pi) = F_n([0, 1])$. This space is disconnected since, for example, there is no path in $F_n([0, 1])$ from $(1/n, 2/n, \ldots, 1)$ to $(1, \ldots, 2/n, 1/n)$. Such a path would consist of n paths in $[0, 1]$, two of which would connect $(1/n, 1)$ to $(1, 1/n)$ by a path in the unit square and such a path must intersect the diagonal, so the path would not be in $F_n([0, 1])$.

Example 6.5. Let X be the circle S^1. Then $\tilde{X}$ may be identified with the line $\mathbb{R}$, and we will show that $F_n(\mathbb{R}, \pi)$ is disconnected.

We first consider the case $n = 2$. In this case we have:

$$F_2(\mathbb{R}, \pi) = \{(x, y) \in \mathbb{R}^2 \mid x - y \notin \mathbb{Z}\},$$

with the subspace topology from $\mathbb{R}^2$, and this is disconnected.

For $n > 2$, there is a natural surjection $f : F_n(\mathbb{R}, \pi) \to F_2(\mathbb{R}, \pi)$ which discards the last $n - 2$ points. Since $F_n(\mathbb{R}, \pi)$ has a disconnected image under f, it must itself be disconnected.

Now we show that $D_n(X)$ has the required connectedness properties for classical covering space theory.

Lemma 6.6. *If X is a connected finite polyhedron, then $D_n(X)$ is connected, locally path connected, and semi-locally simply connected.*

Proof. We have already shown in the proof of Lemma 6.2 that $D_n(X)$ is connected.

For the rest of the proof we will need to look specifically at the topology on $D_n(X)$. For a point x in some metric space with metric d and a real number $\epsilon > 0$, let $B(x, \epsilon)$ be the open ball of radius ϵ around x.

On the space $D_n(X)$ the Hausdorff metric d_H is defined as follows: for $x, y \in D_n(X)$ with $x = \{x_1, \ldots, x_n\}$ and $y = \{y_1, \ldots, y_n\}$, define $d_H(x, y)$ to be the greater of:

$$\min_{1 \le i \le n} \max_{1 \le j \le n} d(x_i, y_j) \qquad \text{and} \qquad \max_{1 \le i \le n} \min_{1 \le j \le n} d(x_i, y_j).$$

Topologizing $D_n(X)$ using the Hausdorff metric is equivalent to the quotient topology from $F_n(X)$ as in Section 1 (compare Proposition 4.2 of [3]).

Now we observe that, when $\epsilon > 0$ is sufficiently small, any n-configuration $x = \{x_1, \ldots, x_n\} \in D_n(X)$ has an ϵ-neighborhood which is naturally identified with n disjoint ϵ-neighborhoods of the

x_i as follows: Let $\epsilon < \frac{1}{2}\min_{i,j} d(x_i, x_j)$, and then the open ball $B_H(x,\epsilon) \subset D_n(X)$ around x of radius ϵ with respect to the Hausdorff metric consists of all configurations y with $d_H(x,y) < \epsilon$. Because of our choice of ϵ, we will have $y \in B_H(x,\epsilon)$ if and only if there is some permutation $\sigma \in \Sigma_n$ such that $d(x_i, y_{\sigma(i)}) < \epsilon$ for each i. Thus

$$B_H(x,\epsilon) = \{\{y_1, \ldots, y_n\} \mid y_{\sigma(i)} \in B(x_i,\epsilon)\},$$

so $B_H(x,\epsilon)$ is naturally identified with the disjoint union of the $B(x_i,\epsilon)$.

It will follow easily that $D_n(X)$ is locally path connected and semilocally simply connected. Let $x = \{x_1, \ldots, x_n\} \in D_n(X)$, and let U be some open set containing x. Then U contains some $B_H(x,\epsilon)$ which is identified with the disjoint union of $B(x_i,\epsilon)$. It suffices to show that $B_H(x,\epsilon)$ is path connected and semilocally simply connected.

If we take some other point $y \in B_H(x,\epsilon)$, then we may label points so that $y = \{y_1, \ldots, y_n\}$ and $d(x_i, y_i) < \epsilon$. Since X is a polyhedron and thus locally path connected, if ϵ is sufficiently small, there will be a path in X from y_i to x_i for each i, and the disjoint union of these paths provides a path in $D_n(X)$ from y to x. Thus $D_n(X)$ is locally path connected.

If we take some loop ω at x with image contained in $B_H(x,\epsilon)$, then because $B_H(x,\epsilon)$ is the disjoint union of the $B(x_i,\epsilon)$, the loop ω naturally splits as a disjoint union of n loops ω_i in X, and we may label them so that ω_i is a loop at x_i with image contained in $B(x_i,\epsilon)$. Since X is a polyhedron, $B(x_i,\epsilon)$ is simply connected and so each ω_i can be contracted to the constant loop at x_i, and thus the union loop γ can be contracted to the constant loop at x. Therefore $D_n(X)$ is locally simply connected, and thus semi-locally simply connected. $\square$

Now we return to the discussion of the relationship between $F_n(\tilde{X},\pi)$ and the universal cover $q : \tilde{D}_n(X) \to D_n(X)$. The group of covering transformations is isomorphic to $\pi_1(D_n(X))$, which is a well studied object: it is the full braid group $B_n(X)$.

When $F_n(\tilde{X},\pi)$ is connected, we have a connected covering $p^n : F_n(\tilde{X},\pi) \to D_n(X)$. Thus by Lemma 6.1 (a) there is a covering map $r : \tilde{D}_n(X) \to F_n(X,\pi)$ with $q = p \circ r$:

$$\begin{tikzcd} \tilde{D}_n(X) \arrow[r, "r"] \arrow[rr, bend right, "q"'] & F_n(\tilde{X},\pi) \arrow[r, "p^n"] & D_n(X) \end{tikzcd} \tag{9}$$

Since the covering group of $F_n(X,\pi)$ is $\pi_1(X)^n \rtimes \Sigma_n$, Lemma 6.1 (b) shows that $\pi_1(X)^n \rtimes \Sigma_n$ occurs naturally as a quotient of $B_n(X)$ by some normal subgroup. Lemma 6.1 (c) gives a homomorphism $\Phi_f : \pi_1(X) \to \pi_1(X)^n \rtimes \Sigma_n$ such that the following diagram commutes:

$$\begin{tikzcd} & B_n(X) \arrow[d, two heads, "\bar{r}"] \\ \pi_1(X) \arrow[ur, "f_\#"] \arrow[r, "\Phi_f"] & \pi_1(X)^n \rtimes \Sigma_n \end{tikzcd} \tag{10}$$

where the vertical arrow is the quotient homomorphism induced by the covering map r.

In terms of the orbit configuration space, this homomorphism Φ_f plays the same role that the induced homomorphism $f_\#$ plays in terms of the universal covering space. In fact we have already made use of Φ_f in the previous sections in the guise of the functions ϕ_i and σ, as the next result demonstrates.

Theorem 6.7. *Let X be a polyhedron not homeomorphic to the circle or interval, $f : X \to D_n(X)$ be a map, and let $\phi_i : \pi_1(X) \to \pi_1(X)$ and $\sigma : \pi_1(X) \to \Sigma_n$ be as in Lemma 2.4. Then Φ_f is given in coordinates as $\Phi_f = (\phi_1, \ldots, \phi_n; \sigma)$.*

Proof. Let $\bar{f}^* = (\bar{f}_1^*, \ldots, \bar{f}_n^*)$ be the basic lift of f, and let $F : \tilde{X} \to \tilde{D}_n(X)$ be a lift of f such that the diagram commutes:

$$\begin{tikzcd} & \tilde{D}_n(X) \arrow[d, "r"] \\ \tilde{X} \arrow[ur, "F"] \arrow[r, "\bar{f}^*"] & F_n(\tilde{X}, \pi) \end{tikzcd}$$

Since $f_\#$ is the induced homomorphism of f on the fundamental group, we can choose base points in $\tilde{D}_n(X)$ so that, for any $\gamma \in \pi_1(X)$ and $\tilde{x} \in \tilde{X}$, we have:

$$F(\gamma\tilde{x}) = f_\#(\gamma)F(\tilde{x}).$$

Applying r to the above gives:

$$\bar{f}^*(\gamma\tilde{x}) = \Phi_f(\gamma)\bar{f}^*(\tilde{x}).$$

We can write the above in coordinates and, using the definition

of ϕ_i and σ, we obtain:

$$\begin{aligned}\Phi_f(\gamma)\bar{f}^*(\tilde{x}) = \bar{f}^*(\gamma\tilde{x}) &= (\bar{f}_1^*(\gamma\tilde{x}), \ldots, \bar{f}_n^*(\gamma\tilde{x})) \\ &= (\phi_1(\gamma)\bar{f}^*_{\sigma_\gamma(1)}(\tilde{x}), \ldots, \phi_n(\gamma)\bar{f}^*_{\sigma_\gamma(n)}(\tilde{x})) \\ &= (\phi_1(\gamma), \ldots, \phi_n(\gamma); \sigma_\gamma) \cdot (\bar{f}_1^*(\tilde{x}), \ldots, \bar{f}_n^*(\tilde{x})) \\ &= (\phi_1, \ldots, \phi_n; \sigma)(\gamma) \cdot \bar{f}^*(\tilde{x})\end{aligned}$$

Since the action of $\pi_1(X)^n \rtimes \Sigma_n$ on $F_n(\tilde{X}, \pi)$ is a covering action, and $\Phi_f(\gamma)\bar{f}^*(\tilde{x}) = (\phi_1, \ldots, \phi_n; \sigma)(\gamma) \cdot \bar{f}^*(\tilde{x})$, this means that $\Phi_f(\gamma) = (\phi_1, \ldots, \phi_n; \sigma)(\gamma)$ as desired. □

In the case where X is a smooth manifold of dimension at least 3, diagrams (9) and (10) will collapse, as follows:

Theorem 6.8. *Let X be a connected polyhedron which is a smooth manifold of dimension at least 3. Then:*

- *$B_n(X)$ is isomorphic to $\pi_1(X)^n \rtimes \Sigma_n$,*
- *$\tilde{D}_n(X)$ is homeomorphic to $F_n(X, \pi)$,*
- *$f_\# = \Phi_f$.*

Proof. [3] It suffices to prove that $B_n(X)$ is isomorphic to $\pi_1(X)^n \rtimes \Sigma_n$. In that case, the covering maps $p^n : F_n(\tilde{X}, \pi) \to D_n(X)$ and $q : \tilde{D}_n(X) \to D_n(X)$ have the same covering groups, and so $F_n(X, \pi)$ and $\tilde{D}_n(X)$ are homeomorphic. Furthermore, the vertical arrow in (10) is the identity homomorphism, and therefore $f_\# = \Phi_f$.

We will use the following algebraic characterization of the semidirect product: If H, G, and N are groups and we have a short exact sequence:

$$1 \to H \xrightarrow{i} G \xrightarrow{j} N \to 1,$$

then $G \cong H \rtimes N$ if the sequence is right-split, i.e. there is a homomorphism $k : N \to G$ such that $j \circ k$ is the identity on N.

Let $P_n(X) \subseteq B_n(X)$ be the subgroup of "pure braids", those for which the underlying permutation of the strands is trivial. There is a well-known short exact sequence ([9], page 16):

$$1 \to P_n(X) \hookrightarrow B_n(X) \xrightarrow{j} \Sigma_n \to 1,$$

where the map from $P_n(X)$ to $B_n(X)$ is the inclusion, and j is the underlying permutation of the braid.

It is a classical theorem of Birman [1, Theorem 1] that when X is a smooth manifold of dimension 3 or higher, the pure braid group

[3] The authors thank Daciberg Gonçalves for suggesting this proof.

$P_n(X)$ is isomorphic to $\pi_1(X)^n$. (Intuitively: since X is high enough dimension, braid strands can pass through each other and so there is no classical braiding of strands wrapping around one another. The only nontrivial elements of $P_n(X)$ arise when the n strands wrap around holes in the space.) Thus our short exact sequence becomes:

$$1 \to \pi_1(X)^n \hookrightarrow B_n(X) \xrightarrow{j} \Sigma_n \to 1,$$

and we need only find a right-inverse $k : \Sigma_n \to B_n(X)$ of j.

Since X is a manifold of dimension 3 or higher, let $U \subset X$ be an open set which is homeomorphic to an open m-ball, $m \geq 3$. Let $i : D_n(U) \to D_n(X)$ be the inclusion, which induces a homomorphism on fundamental groups $i_\# : B_n(U) \to B_n(X)$. Birman's result shows that $P_n(U) \cong \pi_1(U)^n \cong \{1\}$. Then the short exact sequence for U takes the form:

$$1 \to 1 \hookrightarrow B_n(U) \to \Sigma_n \to 1$$

and therefore $B_n(U) \cong \Sigma_n$. Under these isomorphisms, $i_\#$ gives a homomorphism $k : \Sigma_n \to B_n(X)$.

It remains only to show that k is a right-inverse of j, but this is clear: if we begin with a permutation $\sigma \in \Sigma_n$, then $k(\sigma) \in B_n(X)$ is a braid in X which is induced by inclusion from a braid in U with underlying permutation σ. Thus the underlying permutation of $k(\sigma)$ is σ, which is to say that $j(k(\sigma)) = \sigma$ as desired. □

7 The Jiang subgroup

Let $f : X \to D_n(X)$ be an n-valued map. A homotopy $H\colon X \times I \to D_n(X)$ is a *cyclic homotopy of* f if $H(x, 0) = H(x, 1) = f(x)$ for all $x \in X$. A cyclic homotopy of f will lift to a homotopy starting at the basic lifting $\bar{f}^* = (\bar{f}_1^*, \ldots, \bar{f}_n^*)\colon \tilde{X} \to F_n(\tilde{X}, \pi)$ and ending at $(\gamma_1 \bar{f}_{\sigma(1)}, \ldots, \gamma_n \bar{f}_{\sigma(n)})$, where σ is a permutation and $\gamma_i \in \pi_1(X)$. In this way, from each cyclic homotopy we obtain an element $(\gamma_1, \ldots, \gamma_n; \sigma) \in \pi_1(X)^n \rtimes \Sigma_n$. The *Jiang subgroup for n-valued maps* $J_n(\bar{f}^*) \subseteq \pi_1(X)^n \rtimes \Sigma_n$ is the set of all such elements. For $n = 1$, this definition is the same as that of the subgroup $J(\tilde{f})$ of $\pi_1(X)$ introduced by Jiang; see [12], page 30.[4]

Proposition 7.1. *The set $J_n(\bar{f}^*)$ is a subgroup of $\pi_1(X)^n \rtimes \Sigma_n$.*

[4] There is an extension of the Jiang subgroup, due to Gonçalves [7], that applies to maps $f\colon X \to Y$ and is quite different from $J_n(\bar{f})$, which concerns only n-valued maps, that is, the case $Y = D_n(X)$.

Proof. Let $(\alpha_1, \ldots, \alpha_n; \eta), (\beta_1, \ldots, \beta_n; \theta) \in J_n(\bar{f}^*)$. We will show that $J_n(\bar{f}^*)$ is a subgroup by proving that

$$(\alpha_1, \ldots, \alpha_n; \eta)(\beta_1, \ldots, \beta_n; \theta)^{-1} \in J_n(\bar{f}^*).$$

Since $(\alpha_1, \ldots, \alpha_n; \eta) \in J_n(\bar{f}^*)$, there is a cyclic homotopy H of f lifting to n homotopies, of the $\bar{f}_i^*$ to $\alpha_i \bar{f}^*_{\eta(i)}$. Similarly there is a cyclic homotopy K of f lifting to homotopies of $\bar{f}_i^*$ to $\beta_i \bar{f}^*_{\theta(i)}$. Equivalently, K lifts to homotopies of $\beta^{-1}_{\theta^{-1}(i)} \bar{f}^*_{\theta^{-1}(i)}$ to $\bar{f}_i^*$. Applying the permutation η, we see that K lifts to homotopies of $\beta^{-1}_{\eta\theta^{-1}(i)} \bar{f}^*_{\eta\theta^{-1}(i)}$ to $\bar{f}^*_{\eta(i)}$. Then the concatenated homotopy $H * K^{-1}$, where K^{-1} denotes the reverse of K, is a cyclic homotopy of f lifting to homotopies of $\bar{f}_i^*$ to $\alpha_i \beta^{-1}_{\eta\theta^{-1}(i)} \bar{f}^*_{\eta\theta^{-1}(i)}$. Thus

$$(\alpha_1 \beta^{-1}_{\eta\theta^{-1}(1)}, \ldots, \alpha_n \beta^{-1}_{\eta\theta^{-1}(n)}; \eta \circ \theta^{-1}) \in J_n(f),$$

and this element is equal to $(\alpha_1, \ldots, \alpha_n; \eta)(\beta_1, \ldots, \beta_n; \theta)^{-1}$. □

It is natural to ask how the subgroup $J_n(\bar{f}^*)$ depends on the choice of lifting $\bar{f}^*$. An alternative choice will change the Jiang subgroup, but in a predictable way. Any other lifting has the form $\Gamma \bar{f}^*$ for some $\Gamma \in \pi_1(X)^n \rtimes \Sigma_n$, as follows.

Theorem 7.2. *Let $f : X \to D_n(X)$ be an n-valued map, and let $\bar{f}^* : \tilde{X} \to F_n(\tilde{X}, \pi)$ its basic lifting Then $J_n(\Gamma \bar{f}^*)$ is isomorphic to $J_n(\bar{f}^*)$ by an inner automorphism of $\pi_1(X)^n \rtimes \Sigma_n$. In particular $J_n(\Gamma \bar{f}^*) = J_n(\bar{f}^*)^\Gamma$, where the exponent denotes conjugation by Γ.*

Proof. We will show that for each $A = (\alpha_1, \ldots, \alpha_n; \eta) \in J_n(\bar{f}^*)$, we have $\Gamma A \Gamma^{-1} \in J_n(\Gamma \bar{f}^*)$. Since $A \in J_n(\bar{f}^*)$, there is a cyclic homotopy of f which lifts to a homotopy of $\bar{f}^*$ to $A\bar{f}^*$. This same cyclic homotopy, when lifted to start at $\Gamma \bar{f}^*$, will give a homotopy of $\Gamma \bar{f}^*$ to $\Gamma A \bar{f}^* = (\Gamma A \Gamma^{-1}) \Gamma \bar{f}^*$, and thus $\Gamma A \Gamma^{-1} \in J_n(\Gamma \bar{f}^*)$. □

Given $f : X \to D_n(X)$, let $\phi_i : \pi_1(X) \to \pi_1(X)$ and $\sigma : \pi_1(X) \to \Sigma_n$ be the functions given by Lemma 2.4. As in Theorem 6.7, let $\Phi_f : \pi_1(X) \to \pi_1(X)^n \rtimes \Sigma_n$ be given by $\Phi_f(\gamma) = (\phi_1(\gamma), \ldots, \phi_n(\gamma); \sigma_\gamma)$. As long as X is not the circle, this Φ_f is the homomorphism in (10), and if X is a manifold of dimension 3 or greater, this is the induced homomorphism of f on fundamental groups. It is routine to check that Φ_f is a homomorphism using the formulas of Lemma 4.2. (When X is not the circle, it is a homomorphism by (10).)

By Lemma 2.4, for $\bar{f}^* = (\bar{f}_1^*, \ldots, \bar{f}_n^*)$, we have:

$$\begin{aligned}\bar{f}^*(\gamma\tilde{x}) &= (\bar{f}_1^*(\gamma\tilde{x}), \ldots, \bar{f}_n^*(\gamma\tilde{x})) \\ &= (\phi_1(\gamma)\bar{f}^*_{\sigma_\gamma(1)}(\tilde{x}), \ldots, \phi_n(\gamma)\bar{f}^*_{\sigma_\gamma(n)}(\tilde{x})) \\ &= (\phi_1(\gamma), \ldots, \phi_n(\gamma); \sigma_\gamma)\bar{f}^*(\tilde{x}) = \Phi_f(\gamma)\bar{f}^*(\tilde{x}),\end{aligned}$$

and thus $\bar{f}^*\gamma = \Phi_f(\gamma)\bar{f}^*$.

From the definition of $J_n(\bar{f}^*)$, we immediately obtain:

Theorem 7.3. *Let $f : X \to D_n(X)$, and let $\bar{f}^* : \tilde{X} \to F_n(\tilde{X}, \pi)$ be the basic lifting. Then there is a cyclic homotopy from $\bar{f}^*$ to $\bar{f}^*\gamma$ for every $\gamma \in \pi_1(X)$ if and only if $\Phi_f(\pi_1(X)) \subseteq J_n(\bar{f}^*)$.*

The condition that $\Phi_f(\pi_1(X)) \subseteq J_n(\bar{f}^*)$ is the n-valued analogue of the condition in single-valued Nielsen theory that $f_\#(\pi_1(X)) \subseteq J(\tilde{f})$. In the single-valued theory, this condition implies that all fixed point classes have the same index, and this can be used to show in many cases that $N(f) = R(f)$. In the n-valued theory the result is that some, though perhaps not all, of the fixed point classes have the same index.

Lemma 7.4. *Let $f : X \to D_n(X)$ be a map and $\bar{f}^* = (\bar{f}_1^*, \ldots, \bar{f}_n^*)$ be the basic lifting. If $\Phi_f(\pi_1(X)) \subseteq J_n(\bar{f}^*)$, then $p\,\mathrm{Fix}(\gamma\bar{f}_j^*)$ and $p\,\mathrm{Fix}(\delta\bar{f}_j^*)$ have the same fixed point index for each j and any $\gamma, \delta \in \pi_1(X)$.*

Proof. It suffices to show that $p\,\mathrm{Fix}(\bar{f}_i^*)$ and $p\,\mathrm{Fix}(\gamma\bar{f}_i^*)$ have the same index for any i and any $\gamma \in \pi_1(X)$. First observe that the liftings $\Phi_f(\gamma)\bar{f}^*$ and $(\gamma, \ldots, \gamma; 1)\bar{f}^*$ are uniformly conjugate by γ since for each i we have

$$\phi_i(\gamma)\bar{f}^*_{\sigma_\gamma(i)} = \gamma^{-1}\gamma\bar{f}_i^*\gamma.$$

Thus by Corollary 2.6

$$p\,\mathrm{Fix}(\gamma\bar{f}_i^*) = p\,\mathrm{Fix}(\phi_i(\gamma)\bar{f}^*_{\sigma_\gamma(i)}) = p\,\mathrm{Fix}(\bar{f}_i^*\gamma). \tag{11}$$

By Theorem 7.3, since $\Phi_f(\pi_1(X)) \subseteq J_n(\bar{f}^*)$ we have a cyclic homotopy of $\bar{f}_i^*$ to $\bar{f}_i^*\gamma$. Thus, by the homotopy invariance of the fixed point index (Lemma 6.4 of [14]), we have

$$\mathrm{ind}(f, p\,\mathrm{Fix}(\bar{f}_i^*)) = \mathrm{ind}(f, p\,\mathrm{Fix}(\bar{f}_i^*\gamma)) = \mathrm{ind}(f, p\,\mathrm{Fix}(\gamma\bar{f}_i^*)). \qquad \square$$

Proposition 7.5. *Let $f : X \to D_n(X)$ be a map and $\bar{f}^* = (\bar{f}_1^*, \ldots, \bar{f}_n^*)$ be the basic lifting. If $\Phi_f(\pi_1(X)) \subseteq J_n(\bar{f}^*)$, and there is some $\alpha \in \pi_1(X)$ such that $\sigma_\alpha(i) = j$, then $p\,\mathrm{Fix}(\gamma\bar{f}_i^*)$ and $p\,\mathrm{Fix}(\delta\bar{f}_j^*)$*

have the same index for any $\gamma, \delta \in \pi_1(X)$. In particular if, for each $j \in \{2, \ldots, n\}$, there exists $\alpha_j \in \pi_1(X)$ such that $\sigma_{\alpha_j}(1) = j$, then all the fixed point classes of f have the same index and therefore either $N(f) = 0$ or $N(f) = R(f)$.

Proof. The first equality of (11) shows that $p\,\mathrm{Fix}(\gamma \bar{f}_i^*) = p\,\mathrm{Fix}(\phi_i(\gamma)\bar{f}_j^*)$, and then the result of Lemma 7.4 gives:

$$\begin{aligned}\mathrm{ind}(f, p\,\mathrm{Fix}(\gamma \bar{f}_i^*)) &= \mathrm{ind}(f, p\,\mathrm{Fix}(\phi_j(\alpha)\bar{f}_i^*)) \\ &= \mathrm{ind}(f, p\,\mathrm{Fix}(\bar{f}_j^*)) \\ &= \mathrm{ind}(f, p\,\mathrm{Fix}(\delta \bar{f}_j^*)) \qquad \square\end{aligned}$$

The condition of Theorem 7.3 which implies equality of indices is always satisfied for maps on tori:

Theorem 7.6. *Let T^q denote the q-torus, and let $f : T^q \to D_n(T^q)$ be a map. Then $\Phi_f(\pi_1(X)) \subseteq J_n(\bar{f}^*)$.*

Proof. Let $a \in \mathbb{Z}^q \cong \pi_1(T^q)$. We will show that there is a cyclic homotopy of f which lifts to a homotopy of $\bar{f}^*(t)$ to $\bar{f}^*(t+a)$. For $t \in \mathbb{R}^q$ and $s \in [0,1]$, define $\bar{H}(t,s) = \bar{f}^*(t + sa)$. Then $\bar{H}$ is a homotopy of $\bar{f}^*(t)$ to $\bar{f}^*(t+a)$, and each stage of the homotopy is n-valued because $\bar{f}^*(t)$ is n-valued. Let $H(p^q(t), s) = (p^q)^n(\bar{H}(t,s))$. Then we can compute:

$$\begin{aligned}H(p^q(t), 0) &= (p^q)^n(\bar{H}(t,0)) = (p^q)^n \bar{f}^*(t) = f(p^q(t)) \\ H(p^q(t), 1) &= (p^q)^n(\bar{H}(t,1)) = (p^q)^n \bar{f}^*(t+a) \\ &= f(p^q(t+a)) = f(p^q(t))\end{aligned}$$

where the last equality holds because $a \in \mathbb{Z}^q$. Thus H is a cyclic homotopy of f, which lifts to $\bar{H}$, which is a homotopy of $\bar{f}^*(t)$ to $\bar{f}^*(t+a)$. $\square$

To illustrate the previous results, we apply them to the following class of n-valued maps introduced in [4].

For $t = (t_1, \ldots, t_q) \in \mathbb{R}^q$, denote the universal covering space of the torus T^q by $p^q \colon \mathbb{R}^q \to \mathbb{T}^q$ where $p^q(t) = (p(t_1), \ldots, p(t_q))$ for $p(t_j) = \exp(i2\pi t_j)$. We recall that a $q \times q$ integer matrix A induces a map $f_A \colon \mathbb{R}^q/\mathbb{Z}^q = T^q \to T^q$ by

$$f_A(p^q(t)) = p^q(At) = (p(A_1 \cdot t), \ldots, p(A_q \cdot t)),$$

where A_j is the j-th row of A, and that f_A is called a *linear* self-map of T^q.

We define $x = (x_1, \ldots, x_q), y = (y_1, \ldots, y_q) \in \mathbb{R}^q$ to be *congruent mod n*, written $x \equiv y\,(n)$, if $x_j - y_j$ is divisible by n for all $j = 1, \ldots, q$.

For $k \in \mathbb{Z}$, let $\mathbf{k} = (k, k, \ldots, k) \in \mathbb{Z}^q$. Define $f_{n,A}^{(k)} \colon \mathbb{R}^q \to \mathbb{T}^q$ by

$$f_{n,A}^{(k)}(t) = p^q(\frac{1}{n}(At + \mathbf{k})).$$

Theorem 7.7. *([4], Theorem 3.1) A $q \times q$ integer matrix A induces an n-valued map $f_{n,A} \colon T^q \to D_n(T^q)$ defined by*

$$f_{n,A}(p^q(t)) = \{f_{n,A}^{(1)}(t), \ldots, f_{n,A}^{(n)}(t)\}$$

if and only if $A_i \equiv A_j\,(n)$ for all $i, j \in \{1, \ldots, q\}$.

Since $f_{1,A} = f_A : \mathbb{T}^q \to \mathbb{T}^q$, the maps $f_{n,A}$ are called *linear n-valued self-maps of tori.*

The n-valued map $f_{n,A}$ lifts to $\bar{f}_{n,A} = (\bar{f}_1, \ldots, \bar{f}_n) \colon \mathbb{R}^q \to F_n(\mathbb{R}^q, \mathbb{Z}^q)$ where

$$\bar{f}_j(t) = \frac{1}{n}(At + \mathbf{j}).$$

Proposition 7.8. *Let A be a $q \times q$ integer matrix such that $A_i \equiv A_j\,(n)$ for all $i, j \in \{1, \ldots, q\}$ and let $f_{n,A} \colon T^q \to D_n(T^q)$ be the corresponding linear n-valued self-map of T^q, then all the fixed point classes of $f_{n,A}$ have the same index and therefore $R(f_{n,A}) = N(f_{n,A})$.*

Proof. The map $f_{n,A}$ lifts to $\bar{f}_{n,A}^* = (\bar{f}_1^*, \ldots, \bar{f}_n^*) \colon \mathbb{R}^q \to F_n(\mathbb{R}^q, \mathbb{Z}^q)$ where

$$\bar{f}_j^*(t) = \frac{1}{n}(At + \mathbf{j}).$$

Define $\bar{H} \colon = (\bar{h}_1, \ldots, \bar{h}_n) \colon \mathbb{R}^q \times I \to F_n(\mathbb{R}^1, \mathbb{Z}^q)$ by setting

$$\bar{h}_j(t, s) = \frac{1}{n}(At + \mathbf{j} + \mathbf{s}).$$

Then for each $j < n$, the coordinate j of $\bar{H}$ gives a homotopy of $\bar{f}_j^*$ to $\bar{f}_{j+1}^*$. By projecting this homotopy, we have $\operatorname{ind}(f, p\operatorname{Fix}(\bar{f}_j^*)) = \operatorname{ind}(f, p\operatorname{Fix}(\bar{f}_{j+1}^*))$ for each $j < n$.

Now let $p\operatorname{Fix}(\gamma \bar{f}_j^*)$ and $p\operatorname{Fix}(\delta \bar{f}_k^*)$ be any two fixed point classes. By concatenating the homotopies above, $p\operatorname{Fix}(\bar{f}_j^*)$ and $p\operatorname{Fix}(\bar{f}_k^*)$ have the same index. By Theorem 7.6 we have $\Phi_{f_{n,A}}(\pi_1(X)) \subseteq J_n(\bar{f}_{n,A}^*)$, and so we may apply Lemma 7.4. We obtain:

$$\begin{aligned}\operatorname{ind}(f_{n,A}, p\operatorname{Fix}(\gamma \bar{f}_j^*)) &= \operatorname{ind}(f_{n,A}, p\operatorname{Fix}(\bar{f}_j^*)) \\ &= \operatorname{ind}(f_{n,A}, p\operatorname{Fix}(\bar{f}_k^*)) = \operatorname{ind}(f_{n,A}, p\operatorname{Fix}(\delta \bar{f}_k^*)).\end{aligned}$$

Therefore all the fixed point classes have the same index and we conclude that $R(f_{n,A}) = N(f_{n,A})$. □

The Nielsen number of $f_{n,A}$ was calculated in [5] to be $N(f_{n,A}) = n|\det(E - \frac{1}{n}A)|$ where E is the identity matrix, and thus, if $N(f_{n,A})$ is non-zero, we can conclude that $R(f_{n,A}) = n|\det(E - \frac{1}{n}A)|$ also.

8 Split maps

An n-valued map $f\colon X \to D_n(X)$ is *split* if there exist single-valued maps $f_1, \dots, f_n\colon X \to X$ such that $f(x) = \{f_1(x), \dots, f_n(x)\}$ for all $x \in X$. We write $f = \{f_1, \dots, f_n\}$. Schirmer proved ([14], Corollary 7.2) that if $f = \{f_1, \dots, f_n\}$, is split, then the Nielsen number of the n-valued map is relate to the Nielsen number for single-malued maps by

$$N(f) = N(f_1) + \cdots + N(f_n).$$

Theorem 8.1. *Let $f = (f_1, \dots, f_n)$ be a split n-valued map with the associated ϕ_i and σ_α given by Lemma 2.4. Then σ_α is the identity permutation for every α, and each ϕ_i is the induced homomorphism $f_{i\pi}\colon \pi_1(X) \to \pi_1(X)$ of f_i.*

Proof. Let $f = (f_1, \dots, f_n)$ be a splitting of f, and then we can choose $(\bar{f}_1, \dots, \bar{f}_n)$ so that $\bar{f}_i$ is a lifting of f_i. In this case the single-valued covering space theory suffices to show that $\bar{f}_i(\alpha\tilde{x}) = f_{i\pi}(\alpha)\bar{f}_i(\tilde{x})$, where $f_{i\pi}$ is the induced fundamental group homomorphism of f_i. Since ϕ_i and σ_α are defined by the formula $\bar{f}_i(\alpha\tilde{x}) = \phi_i(\alpha)\bar{f}_{\sigma_\alpha(i)}(\tilde{x})$, we have $\sigma_\alpha(i) = i$ and $\phi_i = f_{i\pi}$. □

Since the permutations σ_α are all the identity in the split case, the twisted conjugacy relation has a simpler characterization.

Corollary 8.2. *Let $f = (f_1, \dots, f_n)$ be a split n-valued map with the associated ϕ_i and σ_α given by Lemma 2.4. Then $[\alpha]_i = [\beta]_j$ if and only if $i = j$ and there is some γ such that $\alpha = \gamma^{-1}\beta f_{i\pi}(\gamma)$, where $f_{i\pi}$ is the induced fundamental group homomorphism of f_i.*

The above corollary means that when f splits, no fixed point class $p\,\mathrm{Fix}(\alpha\bar{f}_i^*)$ can equal any fixed point class $p\,\mathrm{Fix}(\beta\bar{f}_j^*)$ when $i \neq j$, and when $i = j$ these fixed point classes are equal exactly when they are equal according to the classical theory of Reidemeister classes of a single-valued map.

Theorem 8.3. *Let $f = (f_1, \dots, f_n)$ be a split n-valued map. Then we have*

$$R(f) = R(f_1) + \cdots + R(f_n),$$

where on the right side, R denotes the classical Reidemeister number of a single-valued map.

Proof. We have already shown that when f is split, we have $[\alpha]_i = [\beta]_j$ if and only if $i = j$ and α and β are classically Reidemeister equivalent by ϕ_i, the induced homomorphism of f_i. Thus the number of Reidemeister classes of f is the total of the number of Reidemeister classes of the various f_i. □

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