FIXED POINTS OF FIBER MAPS OF MONTGOMERY-SAMELSON FIBERINGS OVER INTERVALS

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Abstract

A Montgomery-Samelson (MS) fibering is a fibering with singularities such that the singular fibers are points. A fiber map is a map that preserves fibers and takes singular fibers to singular fibers. For MS fiberings that are the suspension of a map of a space to a point, and hence the base space is an interval, we obtain a formula for the minimum number of fixed points among all fiber maps that are fiber homotopic to a fiber map from the fibering to itself. The formula calculates that minimum number in terms of the minimum number of fixed points among the maps homotopic to self-maps of nonsingular fibers that are restrictions of the fiber map. These minimum numbers can often be computed by Nielsen fixed point theory. We present an example of an MS fibering in which the total space and nonsingular fibers are simply-connected and yet, in contrast to fiber maps of fiberings without singularities, for each natural number k there is a fiber map $f^{[k]}$ such that the minimum number of fixed points of the fiber maps fiber homotopic to $f^{[k]}$ is k.

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1 Introduction

Fixed point theory has been extensively developed for fiber (preserving) maps of Hurewicz fiber spaces, see [6]. The purpose of this paper is to initiate the fixed point theory of fiber maps of fiberings with singularities. The class of fiberings with singularities with which we will work was introduced by Montgomery and Samelson in 1946 [12].

A *Montgomery-Samelson fibering* or, more briefly, an *MS fibering* $\mathfrak{F}_{\mathcal{S}} = (p, E, X; Y : \mathcal{S})$ is an open map $p\colon E \to X$ onto X and a subset

$\mathcal{S}$ of X, called the *singular set*, such that $p\colon p^{-1}(X \setminus \mathcal{S}) \to X \setminus \mathcal{S}$ is a fiber bundle with fiber Y and the restriction of p to $p^{-1}(\mathcal{S})$ is a homeomorphism onto $\mathcal{S}$. The points of $p^{-1}(\mathcal{S})$ are the *singular fibers*. The structure of MS fiberings was studied in [1], [2], [3], and [4].

Extending the concept from the fiber space setting, we define a *fiber map* $f\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}'_{\mathcal{S}'}$ of MS fiberings $\mathfrak{F}_{\mathcal{S}} = (p, E, X; Y : \mathcal{S}), \mathfrak{F}'_{\mathcal{S}'} = (p', E', X'; Y' : \mathcal{S}')$ to be a map of pairs $f\colon (E, p^{-1}(\mathcal{S})) \to (E', p^{'-1}(\mathcal{S}'))$ such that if $e_0, e_1 \in E$ with $p(e_0) = p(e_1)$, then $p'f(e_0) = p'f(e_1)$. Thus, in this setting, a fiber map is not only fiber-preserving but it is also required to take singular fibers to singular fibers. A fiber map f induces a function $\bar{f}\colon (X, \mathcal{S}) \to (X', \mathcal{S}')$ such that $p'f = \bar{f}p$. The function $\bar{f}$ is continuous because f is an open map. Given a fiber map f and $x \in X$, we define $f_x\colon p^{-1}(x) \to p^{'-1}(\bar{f}(x))$ to be the restriction of f.

For $I = [0, 1]$, let $(\mathfrak{F} \times I)_{\mathcal{S}\times I} = (p \times 1, E \times I, X \times I; Y : \mathcal{S} \times I)$ where $(p \times 1)(e, t) = (p(e), t)$. A *fiber homotopy* is a fiber map $H\colon (\mathfrak{F} \times I)_{\mathcal{S}\times I} \to \mathfrak{F}'_{\mathcal{S}'}$. For $t \in I$, maps $h_t\colon E \to E'$ and $\bar{h}_t\colon X \to X'$ are defined by $h_t(e) = H(e, t)$ and $\bar{h}_t(x) = \overline{H}(x, t)$. Fiber maps $f, g\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}'_{\mathcal{S}'}$ are *fiber homotopic* if there exists a fiber homotopy H such that $h_0 = f$ and $h_1 = g$.

For a fiber map $f\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}}$, we denote by $MF(f; \mathfrak{F}_{\mathcal{S}})$ the minimum number of fixed points among all fiber maps g that are fiber homotopic to f. If $\bar{f}\colon X \to X$ such that $\bar{f}(x) = x$ then $f_x\colon p^{-1}(x) \to p^{-1}(x)$ and we let $MF(f_x)$ be the minimum number of fixed points among all maps homotopic to f_x.

There is one example of an MS fibering in [12]: "the circles of latitude and the north and south poles of a two-sphere". Generalizing that example, as in [8], we represent the n-sphere as

$$S^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}\colon x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}$$

and define $p\colon S^n \to [-1, 1]$ by $p(x_0, x_1, \ldots, x_n) = x_0$. We thus obtain an MS fibering with singular set $\mathcal{S} = \{-1, 1\}$ and non-singular fiber $Y = S^{n-1}$. We will call this example $\mathfrak{S}^n$.

We will be concerned with fiber maps of MS fiberings $\mathfrak{F}_{\mathcal{S}} = (p, E, X; Y : \mathcal{S})$ where, as in the examples $\mathfrak{S}^n$, we have $X = [-1, 1]$ and the singular set is $\mathcal{S} = \{-1, 1\}$, its boundary. The only requirement for Y is that it be a Hausdorff space. Since the restriction $p\colon p^{-1}((-1, 1)) \to (-1, 1)$ is a bundle over a contractible space, it is the product bundle $p\colon (-1, 1) \times Y \to (-1, 1)$. Thus E may be viewed as the quotient of $[-1, 1] \times Y$ with $\{-1\} \times Y$ and $\{1\} \times Y$ each identified to a point, that is, $E = \Sigma(Y)$, the (unreduced) suspension of Y. The map $p\colon E \to [-1, 1]$ may be viewed as the suspension of the constant map $p_0\colon Y \to \{0\}$ so we write $\mathfrak{F}_{\mathcal{S}} = \Sigma(p_0)$. In particular, $\mathfrak{S}^n = \Sigma(p_0\colon S^{n-1} \to \{0\})$.

We will obtain a formula for $MF(f; \mathfrak{F}_{\mathcal{S}} = \Sigma(p_0))$ in terms of $MF(f_x)$, for finitely many $x \in [-1,1]$ such that $\bar{f}(x) = x$. If Y is a finite polyhedron without local cut points and is not a 2-manifold, then $MF(f_x) = N(f_x)$, the Nielsen number of f_x [10]. The Nielsen number can be calculated in many cases [11], [5], so the formula is an effective way to calculate $MF(f; \mathfrak{F}_{S})$.

In the next section, we present examples of specific fiber maps $f\colon \mathfrak{S}^2 \to \mathfrak{S}^2$ that illustrate some significant features of fiber maps of MS fiberings. In particular, an example illustrates the fact that, in contrast to fiber bundle theory, if $f\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}} = (p, E, X; Y : \mathcal{S})$ is a fiber map inducing $\bar{f}\colon X \to X$ and $\bar{g}$ is homotopic to $\bar{f}$, there may not be a fiber map $g\colon E \to E$ fiber homotopic to f that induces $\bar{g}$. In Section 3, given a fiber map $f\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}} = \Sigma(p_0)$, we will define a homotopy of $\bar{f}\colon [-1,1] \to [-1,1]$ to a map $\bar{g}$ that is in a more convenient form and do it in such a way that there is fiber map g fiber homotopic to f that does induce $\bar{g}$. A construction in Section 4 fiber homotopes a fiber map f to a fiber map g that is in a special form and also induces the map $\bar{g}$ of Section 3. The computation of $MF(f; \mathfrak{F}_{\mathcal{S}} = \Sigma(p_0))$ is carried out in Section 5 by demonstrating that the number of fixed points of the map g of Section 4 is minimal among all fiber maps homotopic to f. In Section 6 we apply the formula of Section 5 to fiber maps of some MS fiberings over intervals. In particular we demonstrate that there is the following important difference between the fixed point theory of fiber maps of fiber spaces and fiber maps of fiberings with singularities. If $p\colon E \to X$ is a fiber space such that E, X and the fibers are simply-connected manifolds and $f\colon E \to E$ is a fiber map, then Theorem 8.2 of [7] implies that f is fiber homotopic to a fiber map with at most one fixed point. We show in Section 6 that even though for $\mathfrak{S}^n = \Sigma(p_0\colon S^{n-1} \to \{0\})$ with $n \geq 3$ we have $E = S^n, X = [-1,1]$ and $Y = S^{n-1}$ all simply-connected, for each integer $k \geq 1$ there is a fiber map $f^{[k]}\colon \mathfrak{S}^n \to \mathfrak{S}^n$ such that $MF(f^{[k]}; \mathfrak{S}^n) = k$.

2 Some fiber maps of $\mathfrak{S}^2$

We will illustrate some important features of fiber maps of MS fiberings $\mathfrak{F}_{\mathcal{S}} = \Sigma(p_0)$ by the following examples on $\mathfrak{S}^2$.

Write a point e in the sphere S^2 in spherical coordinates as $e = (r, \theta, \phi)$ where r is the radius, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ the elevation and $\phi \in \mathbb{R}$ the azimuth. We will take S^2 to be the unit sphere, so we suppress the radius $r = 1$ and write $e = (\theta, \phi)$ if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. For the poles, we set $\nu = (\frac{\pi}{2}, [\phi])$ and $\sigma = (-\frac{\pi}{2}, [\phi])$, where $[\phi]$ denotes the equivalence class consisting of all $\phi \in \mathbb{R}$. Defining the projection $p\colon S^2 \to [-1, 1]$ by $p(e) = p(\theta, \phi) = \sin(\theta)$, we obtain the MS fibering $\mathfrak{S}^2$.

We write the unit sphere S^2 as the union of hemispheres $S^2 = D^2_- \cup D^2_+$ where, in spherical coordinates,

$$D^2_+ = \{e = (\theta, \phi) \in S^2 \colon 0 \leq \theta \leq \frac{\pi}{2}\}$$

and

$$D^2_- = \{e = (\theta, \phi) \in S^2 \colon -\frac{\pi}{2} \leq \theta \leq 0\}.$$

Then, for integers a, b, we define $f\colon S^2 \to S^2$ by

$$f(e) = f(\theta, \phi) = \begin{cases} (2\theta - \frac{\pi}{2}, b\phi) & \text{if } e \in D^2_+ \\ (-2\theta - \frac{\pi}{2}, a\phi) & \text{if } e \in D^2_- \end{cases}$$

which is a continuous fiber map $f\colon \mathfrak{S}^2 \to \mathfrak{S}^2$ because if $e \in D^2_- \cap D^2_+$ then $f(e) = \sigma$.

To compute the degree of f, we will write f as the composition of two maps. Define a translation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ in cartesian coordinates by setting $T(x, y, z) = (x, y, z - 2)$ and note that $T(\nu) = \sigma$. Maps $\delta_+\colon D^2_+ \to S^2$ and $\delta_-\colon D^2_- \to S^2$ are defined by

$$\delta_+(e) = \delta_+(\theta, \phi) = (2\theta - \frac{\pi}{2}, b\phi)$$

and

$$\delta_-(e) = \delta_-(\theta, \phi) = (2\theta + \frac{\pi}{2}, a\phi).$$

We define a map $F\colon S^2 \to S^2 \cup T(S^2)$ by setting

$$F(e) = \begin{cases} \delta_+(e) & \text{if } e \in D^2_+ \\ T\delta_-(e) & \text{if } e \in D^2_-. \end{cases}$$

Let $\mathbf{1} \in H_2(S^2) \cong \mathbf{Z}$ be a generator, then $F_*(\mathbf{1}) = (b\mathbf{1}, aT_*(\mathbf{1})) \in H_2(S^2 \cup T(S^2)) \cong \mathbf{Z} \oplus \mathbf{Z}$. Now define an orientation-reversing homeomorphism $h\colon T(S^2) \to S^2$, in cartesian coordinates, by $h(x, y, z) = (x, y, -z - 2)$, then $h_*(T_*(\mathbf{1})) = -\mathbf{1}$. Noting that $h(\sigma) = \sigma$, we extend to $h\colon S^2 \cup T(S^2) \to S^2$ by letting h be the identity on S^2. Then $f = hF$ and we see that $f_*(\mathbf{1}) = b\mathbf{1} - a\mathbf{1} = (b - a)\mathbf{1}$ so we conclude that f is a map of degree $b - a$.

Consider $\bar{f}\colon [-1, 1] \to [-1, 1]$ and observe that $\bar{f}^{-1}(1) = \{-1, 1\}$ and $\bar{f}^{-1}(-1) = \{0\}$ so that $\bar{f}^{-1}(\mathcal{S}) = \{-1, 0, 1\}$ where $\bar{f}(-1) =$

$-\bar{f}(0)$ and $\bar{f}(0) = -\bar{f}(1)$. An orientation of S^2 induces an orientation of $p^{-1}(x)$ for all $x \in (-1,1)$. Therefore, if $x, \bar{f}(x) \in (-1,1)$, then the degree $\deg(f_x)$ of the map $f_x \colon p^{-1}(x) \to p^{-1}(\bar{f}(x))$ is well-defined. In particular, if $-1 < x < 0$, then $\deg(f_x) = a$ and if $0 < x < 1$, then $\deg(f_x) = b$. We record this information as an ordered pair of integers $\tau(f) = (a,b)$ that we will call the *type* of f.

If there is a fiber map $g \colon S^2 \to S^2$ fiber homotopic to f such that $\bar{g}^{-1}(-1) = \emptyset$, then $g(S^2) \subseteq S^2 \setminus \{\sigma\}$ is therefore inessential so $\deg(f) = \deg(g) = 0$ and thus $a = b$. Conversely, if $a = b$, then the fiber homotopy $H \colon \mathfrak{S}^2 \times I \to \mathfrak{S}^2$ defined by

$$H(\theta,\phi,t) = (2(1-t)|\theta| + \pi t - \tfrac{\pi}{2}, a\phi)$$

is a homotopy between f and the constant map of S^2 to ν. Since any fiber map fiber homotopic to f must fix ν and the constant map has a single fixed point, we conclude that $MF(f;\mathfrak{S}^2) = 1$ if $a = b$.

Now suppose that $a \neq b$. The nonzero degree map $f \colon S^2 \to S^2$ illustrates an important difference between the behavior of fiber maps of fiberings with singularities and fiber maps of nonsingular fiber spaces. The homotopy lifting property of fiber spaces implies that if $p \colon E \to X$ is a fiber space, $f \colon E \to E$ is a fiber map inducing $\bar{f} \colon X \to X$ and $\bar{g} \colon X \to X$ is homotopic to $\bar{f}$, then there is a fiber map $g \colon E \to E$, fiber homotopic to f, that induces $\bar{g}$. The map $\bar{f} \colon [-1,1] \to [-1,1]$ induced by the example map f is homotopic, relative to $\mathcal{S}$, to the constant map $\bar{g}$ of $[-1,1]$ to 1. But the constant map g of S^2 to ν is of degree zero, so g is not homotopic to f.

Let $H \colon (S^2,\nu,\sigma) \to (S^2,\nu,\nu)$ be a fiber homotopy such that $h_0 = f$. Although $\bar{f}^{-1}(\mathcal{S})$ is finite, the map $g = h_1$ need not have this property. For example the fiber homotopy defined by

$$H(\theta,\phi,t) = \begin{cases} f(\theta,\phi) & \text{if } 0 \le \theta \le \frac{\pi}{2}, \\ (\frac{4\theta}{t-2} - \frac{\pi}{2}, a\phi) & \text{if } \frac{t\pi}{4} - \frac{\pi}{2} \le \theta \le 0, \\ \nu & \text{if } -\frac{\pi}{2} \le \theta \le \frac{t\pi}{4} - \frac{\pi}{2} \end{cases}$$

has $\bar{g}^{-1}(\mathcal{S}) = \{[-1,-\frac{1}{2}], 0, 1\}$. We observe that

$$\deg(g_x) = \begin{cases} 0 & \text{if } -1 < x < -\frac{1}{2}, \\ a & \text{if } -\frac{1}{2} < x < 0, \\ b & \text{if } 0 < x < 1. \end{cases}$$

Noting that $\bar{g}(-\frac{1}{2}) = -\bar{g}(0)$ and $\bar{g}(0) = -\bar{g}(1)$ but $\bar{g}(-1) = \bar{g}(-\frac{1}{2})$, we will not record the degree 0 so the type of g is $\tau(g) = (a,b)$ also.

Even ignoring the degree of regions whose endpoints map to the same pole, the type of a fiber map is not a homotopy invariant. For the fiber homotopy defined by

$$K(\theta,\phi,t)=\begin{cases} g(\theta,\phi) & \text{if } -\frac{\pi}{4}\leq\theta\leq\frac{\pi}{2},\\ (8t\theta+\frac{\pi}{2}+2\pi t, c\phi) & \text{if } -\frac{3\pi}{8}\leq\theta\leq-\frac{\pi}{4},\\ (-8t\theta+\frac{\pi}{2}-4\pi t, c\phi) & \text{if } -\frac{\pi}{2}\leq\theta\leq-\frac{3\pi}{8}\end{cases}$$

we have $k_0 = g$, the map defined by the previous fiber homotopy, and $\bar{k}_1^{-1}(\mathcal{S}) = \{-1, -\frac{3}{4}, -\frac{1}{2}, 0, 1\}$ with type the 4-tuple $\tau(k_1) = (c, c, a, b)$. The fiber homotopy K illustrates a fact that will be established in Section 4 in a more general setting. Equal degrees in adjacent positions in the type of a fiber map, such as (c, c) in $\tau(k_1)$, can be cancelled by a fiber homotopy, as in this case to k_0 for which $\tau(k_0) = (a, b)$.

The example map f fixes $\nu \in S^2$ as well as the fixed points of $f_{-\frac{1}{3}}$, which are $|a-1|$ in number. By a theorem is Section 5, the facts that $\tau(f) = (a, b)$ with $a \neq b$ and that $\bar{f}(1) = 1$ imply that $MF(f;\mathfrak{S}^2) = |a-1| + 1$ if $a \neq b$ (see Prop. 6.1).

3 Skeletal interval maps

Let $\mathfrak{F}_{\mathcal{S}}$ be an MS fibering over an interval, that is, $\mathfrak{F}_{\mathcal{S}} = \Sigma(p_0\colon Y \to \{0\})$ so $\mathcal{S} = \{-1, 1\}$. Then $E = [-1, 1] \times Y/\sim$ where $(-1, y) \sim (-1, y')$ and $(1, y) \sim (1, y')$ for all $y, y' \in Y$, so write $e \in E$ as $e = (x, [y])$ where $[y] = y$ unless $x \in \mathcal{S}$. Thus we may write the fiber map $f\colon E \to E$ in the form $f(e) = f(x, [y]) = (\bar{f}(x), [\pi_2 f_x(e))])$ where π_1, π_2 are the projections of $[-1, 1] \times Y$ onto its factors.

For a fiber map f inducing $\bar{f}\colon ([-1, 1], \mathcal{S}) \to ([-1, 1], \mathcal{S})$, a homotopy $\overline{H}\colon ([-1, 1], \mathcal{S}) \times I \to ([-1, 1], \mathcal{S})$ such that $\bar{h}_0 = \bar{f}$ will be called *liftable to* f if there is a fiber homotopy $H\colon (F \times I)_{\mathcal{S}\times I} \to \mathfrak{F}_{\mathcal{S}}$ inducing $\overline{H}$ such that $h_0 = f$. An example of the previous section demonstrated that not all homotopies of $[-1, 1]$ relative to $\mathcal{S}$ are liftable. If $\overline{H}$ is liftable to f, we say that $\bar{h}_0 = \bar{f}$ and $\bar{h}_1 = \bar{g}$ are *liftably homotopic*. The purpose of this section is to prove that, for any fiber map f on $\mathfrak{F}_{\mathcal{S}}$, the induced map $\bar{f}$ is liftably homotopic to a map that has the following convenient form.

Definition. *A map* $\bar{g}\colon ([-1, 1], \mathcal{S}) \to ([-1, 1], \mathcal{S})$ *is* skeletal *if*

$$\bar{g}^{-1}(\mathcal{S}) = \{-1 = \rho_0, \rho_1, \ldots, \rho_m = 1\},$$

$\bar{g}(\rho_{j-1}) \neq \bar{g}(\rho_j)$ *and* $\bar{g}$ *is linear on* $[\rho_{j-1}, \rho_j]$*, for all* $j = 1, \ldots, m$*. Call the* ρ_j *the* skeletal vertices *of* $\bar{g}$*.*

For the fiber map f inducing $\bar{f}\colon ([-1,1],\mathcal{S}) \to ([-1,1],\mathcal{S})$, we set $\rho_0 = -1$ and define $\rho_j, \lambda_j \in [-1,1]$ for $j \geq 1$ by

$$\rho_j = \min\{\bar{f}^{-1}((-1)^j \bar{f}(-1)) \cap [\rho_{j-1}, 1]\}$$

and

$$\lambda_j = \max\{\bar{f}^{-1}((-1)^{j-1} \bar{f}(-1)) \cap [\rho_{j-1}, \rho_j]\}.$$

Let $A_j = (\lambda_j, \rho_j)$ then $\bar{f}(A_j) \subseteq (-1,1)$ by the definition of ρ_j and λ_j. Since, for each j, the value of $\rho_j - \lambda_j$ is at least as large as the distance between the disjoint closed sets $\bar{f}^{-1}(-1)$ and $\bar{f}^{-1}(1)$, there are only finitely many intervals $A_1, \ldots, A_m$. If $\bar{f}$ is skeletal, then $\lambda_j = \rho_{j-1}$ and therefore $A_j = (\rho_{j-1}, \rho_j)$.

Let $\nu = p^{-1}(1)$ and $\sigma = p^{-1}(-1)$ which, as in $\mathfrak{S}^2$, we will call the *poles* of $E = \Sigma(Y)$.

Theorem 3.1. *Given a fiber map* $f\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}} = \Sigma(p_0)$, *the induced map* $\bar{f}\colon ([-1,1],\mathcal{S}) \to ([-1,1],\mathcal{S})$ *is liftably homotopic to a skeletal map* $\bar{g}\colon [-1,1] \to [-1,1]$ *with skeletal vertices* $\{\rho_j\}$ *where* $\rho_0 = -1$ *and*

$$\rho_j = \min\{\bar{f}^{-1}((-1)^j \bar{f}(-1)) \cap [\rho_{j-1}, 1]\}.$$

Proof. We will prove the theorem by constructing a fiber homotopy that induces a homotopy of $\bar{f}$ to the required skeletal map. For each j, we will define the fiber homotopy in such a way that $\bar{g}(\rho_j) = \bar{f}(\rho_j)$ and thus $\bar{g}(\rho_{j-1}) \neq \bar{g}(\rho_j)$ by the definition of the ρ_j. The construction of the fiber homotopy will be carried out in three steps.

For the first step, set $\lambda_{m+1} = 1$. By the definition of the ρ_j and λ_j, the image of $p^{-1}([\rho_j, \lambda_{j+1}])$ under f is in the complement of one of the poles. We will consider just the case $f(p^{-1}([\rho_j, \lambda_{j+1}])) \subseteq E - \{\sigma\}$ because the other case is no different. Define a fiber homotopy $H^{(j)}\colon p^{-1}([\rho_j, \lambda_{j+1}]) \times I \to E = [-1,1] \times Y/\sim$ by

$$H^{(j)}(e,t) = H^{(j)}((x,[y]),t) = ((1-t)\bar{f}(x) + t, [\pi_2 f_x(e)])$$

then $h_0 = f$ and $h_1(p^{-1}([\rho_j, \lambda_{j+1}])) = \nu$. The continuity of $H^{(j)}$ follows from the facts that $\bar{f}(x) \neq -1$ for $\rho_j \leq x \leq \lambda_{j+1}$ and that if $(1-t)\bar{f}(x) + t = 1$, for $t < 1$, then $\bar{f}(x) = 1$. Since $x = \rho_j$ and $x = \lambda_{j+1}$ imply that $f(e) = \nu$, then $H^{(j)}(e,t) = \nu$ for all t if $e \in p^{-1}(\rho_j)$ or $e \in p^{-1}(\lambda_{j+1})$. Therefore, we may combine the $H^{(j)}$ on $p^{-1}([\rho_j, \lambda_{j+1}])$ with the constant homotopy of f on $p^{-1}([\lambda_j, \rho_j])$ to define a fiber homotopy $H\colon E \times I \to E$ such that h_1 takes each $p^{-1}([\rho_j, \lambda_{j+1}])$ to one of the poles.

In the next step, we again consider only the case $f(p^{-1}([\rho_j, \lambda_{j+1}]) \subseteq E - \{\sigma\}$. For $t \in I$ we define $\bar{k}_t\colon [t\rho_j + (1-t)\lambda_{j+1}, \rho_{j+1}] \to [-1,1]$

for $j < m$ by

$$\bar{k}_t(x) = \bar{h}_1\left(\frac{(\rho_{j+1} - \lambda_{j+1})x + t(\lambda_{j+1} - \rho_j)\rho_{j+1}}{\rho_{j+1} - (t\rho_j + (1-t)\lambda_{j+1})}\right).$$

Note that $\bar{k}_1(\rho_j) = \bar{h}_1(\lambda_{j+1}) = 1$ whereas if $x \in (\rho_j, \lambda_{j+1})$ then $\bar{k}_1(x) = \bar{h}_1(x^*)$ for some $x^* \in (\lambda_{j+1}, \rho_{j+1})$ so $\bar{k}_1(x) \neq \pm 1$. Therefore, defining $K^{(j)} \colon p^{-1}([\rho_j, \rho_{j+1}]) \times I \to E$ by

$$K^{(j)}(e,t) = K^{(j)}((x,[y]),t) =$$

$$\begin{cases} \nu & \text{if } \rho_j \leq x \leq t\rho_j + (1-t)\lambda_{j+1} \\ (\bar{k}_t(x), [\pi_2 f_{\bar{k}_t(x)}(e)]) & \text{if } t\rho_j + (1-t)\lambda_{j+1} \leq x \leq \rho_{j+1} \end{cases}$$

we obtain a fiber homotopy from $k_0 = h_1$ to a fiber map k_1 such that $\bar{k}^{-1}(\mathcal{S}) \cap [\rho_j, \lambda_{j+1}] = \{\rho_j\}$ for $j < m$. By a similar construction on $p^{-1}([\rho_m, 1])$, we have $\bar{k}_1^{-1}(\mathcal{S}) \cap [\rho_m, 1] = \{1\}$. Combining the fiber homotopies $K^{(j)}$, we obtain a fiber homotopy $K \colon E \times I \to E$ such that $k_0 = h_1$ and $\bar{k}_1^{-1}(\mathcal{S}) = \{-1, \rho_1, \ldots, \rho_m, 1\}$.

As the final step, redefining ρ_{m+1} as 1, for each j define $\bar{g}_j \colon [\rho_j, \rho_{j+1}] \to [-1,1]$ by

$$\bar{g}_j(x) = \bar{k}_1(\rho_j)\left(\frac{\rho_{j+1} + \rho_j - 2x}{\rho_{j+1} - \rho_j}\right).$$

Then, for each j, define a fiber map $L^{(j)} \colon p^{-1}([\rho_j, \rho_{j+1}]) \times I \to E$ by

$$L^{(j)}(e,t) = L^{(j)}((x,[y]),t) = ((1-t)\bar{k}_1(x) + t\bar{g}_j(x), [\pi_2 f_{\bar{k}_1(x)}(e)]).$$

Since $L^{(j)} = L^{(j+1)}$ on $p^{-1}(\rho_{j+1})$, the $L^{(j)}$ define a fiber homotopy $L \colon E \times I \to E$ from $\ell_0 = k_1$ to a map $g = \ell_1$ with the required properties. $\square$

4 Skeletal fiber maps

Let $f \colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}} = \Sigma(p_0 \colon Y \to \{0\})$ be a fiber map and choose $x_j \in A_j = (\lambda_j, \rho_j)$. Define $f^*_{x_j} \colon Y \to Y$ by $f^*_{x_j}(y) = \pi_2 f(x_j, y)$, then the homotopy class $[f^*_{x_j}] \in [Y, Y]$ of $f^*_{x_j}$ is independent of the choice of x_j. The m-tuple of homotopy classes $\tau(f) = ([f^*_{x_1}], \ldots, [f^*_{x_m}])$ will be called the *type* of f. This definition extends the concept of type in Section 2 because $[S^1, S^1] \cong \mathbb{Z}$. A type $\tau(f) = ([f^*_{x_1}], \ldots, [f^*_{x_m}])$ is said to be *reduced* if $[f^*_{x_j}] \neq [f^*_{x_{j+1}}]$ for all j. The *reduced type* $\tau_0(f)$ of f is the maximal reduced ordered subset of $\tau(f)$. If f is homotopic to a constant map, then $\tau(f) = \emptyset$.

Definition. *A fiber map* $f\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}}$ *is* skeletal *if the induced map* $\bar{f}\colon [-1,1] \to [-1,1]$ *is skeletal and, for* $\{\rho_j\}$ *the skeletal vertices of* $\bar{f}$*, the restriction* $f_j\colon A_j \times Y = (\rho_{j-1}, \rho_j) \times Y \to (-1,1) \times Y$ *of* f *is a product* $f_j = \bar{f}_j \times f^*_{x_j}$ *for some* $x_j \in A_j$*.*

Lemma 4.1. *A fiber map* $f\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}}$ *is fiber homotopic to a skeletal fiber map* g *such that* $\tau(f) = \tau(g)$*.*

Proof. By Theorem 3.1, we may assume that $\bar{f}\colon [-1,1] \to [-1,1]$ is skeletal with skeletal vertices $\{\rho_j\}$. Let $f_j\colon A_j \times Y \to (-1,1) \times Y$ be the restriction of f. Choose some $x_j \in A_j$ and define $h_{j,t}\colon A_j \times Y \to (-1,1) \times Y$ by

$$h_{j,t}(x,y) = (\bar{f}(x), f^*_{(1-t)x+tx_j}(y))$$

then $h_{j,0} = f_j$ and $h_{j,1} = \bar{f}_j \times f^*_{x_j}$. Extend $h_{j,t}$ to $h_t\colon E \to E$ by setting $h_t(e) = f(e)$ for all t if $e \in p^{-1}(\rho_j)$ for some j. Note that $\bar{h}_t = \bar{f}$ for all t. To prove that h_t is a homotopy, we must show that $H\colon E \times I \to E$ defined by $H(e,t) = h_t(e)$ is continuous at $(e, t_0) \in E \times I$ such that $p(e) = \rho_j$, since H is clearly continuous elsewhere. We suppose that $\bar{f}(\rho_j) = 1$; the case $\bar{f}(\rho_j) = -1$ is the same. Then $f(e) = \nu$ since $pf = \bar{f}p$. Given a neighborhood U of ν in E, we know that $\bar{f}^{-1}(p(U))$ is open in $[-1,1]$ because p is an open map and $\bar{f}$ is continuous. By the continuity of $p \times 1\colon E \times I \to [-1,1] \times I$, there is a neighborhood V of (e, t_0) in $p^{-1}(\bar{f}^{-1}(p(U)) \times I$ such that $H(V) \subseteq U$, and this establishes the continuity of the homotopy h_t. Then $h_0 = f$ and $g = h_1$ is a skeletal fiber map with skeletal vertices $\{\rho_j\}$. Moreover, $g^*_{x_j} = f^*_{x_j}$ for all j so $\tau(g) = \tau(f)$. □

A fiber map f is *reduced* if its type is reduced, that is, if $\tau(f) = \tau_0(f)$.

Theorem 4.1. *A fiber map* $f\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}}$ *is fiber homotopic to a reduced skeletal fiber map.*

Proof. By Lemma 4.1, we may assume that f is a skeletal fiber map of type $\tau(f) = ([f^*_{x_1}], \dots, [f^*_{x_m}])$. It is sufficient to prove that if $[f^*_{x_{j_o}}] = [f^*_{x_{j_o+1}}]$, then there is a skeletal fiber map g fiber homotopic to f that is of type

$$([f^*_{x_1}], \dots, [f^*_{x_{j_o-1}}], [f^*_{x_{j_o+2}}], \dots, [f^*_{x_m}]).$$

We may assume that $f_j = \bar{f}_j \times f^*_{x_j}$ for all j and that there is a homotopy $h_{j_0,t}\colon Y \to Y$ such that $h^*_{j_o,0} = f^*_{x_{j_o}}$ and $h^*_{j_o,1} = f^*_{x_{j_o+1}}$. Define a fiber homotopy h_t by

$$h_t(e) = \begin{cases} (\bar{f}(x), h^*_{j_o,t}(y)) & \text{if } e = (x,y) \in p^{-1}(A_j) \\ f(e) & \text{otherwise} \end{cases}$$

then $h_0 = f$ and the skeletal map h_1 has the property that $h_1(x, y) = (\bar{f}(x), f^*_{j_o}(y))$ for all (x, y) in $p^{-1}(A_{j_o} \cup A_{j_o+1})$. The argument for the continuity of the homotopy is the same as in the proof of Lemma 4.1.

Suppose $\bar{f}(\rho_{j_o-1}) = \bar{f}(\rho_{j_o+1}) = 1$, then define $\bar{k}_t \colon A_{j_o} \cup A_{j_o+1} \to [-1, 1]$ by

$$\bar{k}_t(x) = \begin{cases} \frac{(2t-2)x+\rho_{j_o}+(1-2t)\rho_{j_o-1}}{\rho_{j_o}-\rho_{j_o-1}} & \text{if } x \in A_{j_o} \\ \\ \frac{(2-2t)x+(2t-1)\rho_{j_o+1}-\rho_{j_o}}{\rho_{j_o+1}-\rho_{j_o}} & \text{if } x \in A_{j_o+1} \end{cases}$$

and define a fiber homotopy k_t by

$$k_t(e) = \begin{cases} (\bar{k}_t(x), f^*_{j_o}(y)) & \text{if } e = (x, y) \in p^{-1}(A_{j_o} \cup A_{j_o+1}) \\ f(e) & \text{otherwise.} \end{cases}$$

The argument for the continuity of the homotopy is again as in the proof of Lemma 4.1. Since $\bar{k}_1(x) = 1$ for all $x \in [\rho_{j_o-1}, \rho_{j_o+1}]$, the fiber map k_1 is of the required type. By Lemma 4.1, there is a skeletal fiber map g fiber homotopic to k_1 that is of the same type. The case $\bar{f}(\rho_{j_0-1}) = \bar{f}(\rho_{j_0+1}) = -1$ is essentially the same. $\square$

5 The Minimum Fixed Point Formula

Let $g \colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}} = \Sigma(p_0 \colon Y \to \{0\})$ be a fiber map of type $\tau(g) = ([g^*_{x_1}], \ldots, [g^*_{x_m}])$. Define $MF_0[g^*_{x_k}]$ by setting $MF_0[g^*_{x_1}] = 1$ if $\bar{g}(-1) = -1$, $MF_0[g^*_{x_m}] = 1$ if $\bar{g}(1) = 1$ and, otherwise, $MF_0[g^*_{x_k}] = MF[g^*_{x_k}]$, the minimum number of fixed points among all maps homotopic to $g^*_{x_k} \colon Y \to Y$.

Lemma 5.1. *A fiber map* $g \colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}}$ *of type* $\tau(g) = ([g^*_{x_1}], \ldots, [g^*_{x_m}])$*, has at least* $\sum_{k=1}^{m} MF_0[g^*_{x_k}]$ *fixed points.*

Proof. If $\bar{g}(x) = x$ for some $x \in A_k$, then choose x_k to be that point and we have $g_{x_k} \colon p^{-1}(x_k) \to p^{-1}(x_k)$. Since $MF[g_{x_k}] = MF[g^*_{x_k}]$, then g_{x_k} has at least $MF_0[g^*_{x_k}]$ fixed points. There is a fixed point of $\bar{g}$ in $A_k = (\lambda_k, \rho_k)$ except possibly in the cases that $\lambda_k = -1$ and $\bar{g}(-1) = -1$ or $\rho_k = 1$ and $\bar{g}(1) = 1$. Thus in these cases there are still at least $MF_0[g^*_{x_k}] = 1$ fixed points. $\square$

Theorem 5.1. *A fiber map* $f \colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}}$ *with reduced type* $\tau_0(f) = ([f^*_{x_1}], \ldots, [f^*_{x_{m_0}}])$ *is fiber homotopic to a fiber map with* $\sum_{j=1}^{m_0} MF_0[f^*_{x_j}]$ *fixed points.*

Proof. By Theorem 4.1 we may assume that f is a reduced skeletal fiber map of type $\tau_0(f)$. For $j = 2, \ldots, m_0 - 1$, there is exactly one $x_j \in A_j$ such that $\bar{f}(x_j) = x_j$ and we choose that point x_j to define $f^*_{x_j}$. By definition there is a homotopy $h^* \colon Y \to Y$ such that $h^*_0 = f^*_{x_j}$ and h^*_1 has $MF[f^*_{x_j}]$ fixed points. Since f is skeletal, $f_j = \bar{f}_j \times f^*_{x_j}$. Defining $h^*_{j,t} = \bar{f}_j \times h^*_t \colon A_j \times Y \to (-1,1) \times Y$, then $h_{j,t} \colon A_j \times Y \to A_j \times Y$ has the properties that $h_{j,0} = f_j$ and $h_{j,1}$ has $MF[f^*_{x_j}]$ fixed points on $A_j \times Y$. If $A_j = (\rho_{j-1}, \rho_j)$ such that $\rho_{j-1} = -1$ and $\bar{f}(-1) = 1$ or $\rho_j = 1$ and $\bar{f}(1) = -1$, there is a single fixed point of $\bar{f}$ on A_j and the same construction gives $h_{j,1}$ with $MF[f^*_{x_j}]$ fixed points. The only cases in which there is no fixed point of $\bar{f}$ on A_j is $j = 1$ with $\bar{f}(-1) = -1$ and $j = m_0$ with $\bar{f}(1) = 1$, so each of these cases contributes $MF_0[f^*_{x_j}] = 1$ fixed point. □

The following theorem presents the minimum fixed point formula promised in the introduction.

Theorem 5.2. *If $f \colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}}$ is a fiber map with reduced type $\tau_0(f) = ([f^*_{x_1}], \ldots, [f^*_{x_{m_0}}])$ and g is fiber homotopic to f then $\tau(g)$ contains $[f^*_{x_j}]$ for all j. Consequently*

$$MF(f; \mathfrak{F}_{\mathcal{S}}) = \sum_{j=1}^{m_0} MF_0[f^*_{x_j}].$$

Proof. Let $h_t \colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}}$ be a fiber homotopy such that $h_0 = f$ and $h_1 = g$. Define $\rho_0(t) = -1$ for all t and, noting that $\bar{h}_t(-1) = \bar{f}(-1)$ and $\bar{h}_t(1) = \bar{f}(1)$, we extend the notation of Section 3 by defining, for $j \geq 1$,

$$\rho_j(t) = \min\{\bar{h}_t^{-1}((-1)^j \bar{f}(-1) \cap [\rho_{j-1}(t), 1]\},$$

$$\lambda_j(t) = \max\{\bar{h}_t^{-1}((-1)^{j-1} \bar{f}(-1)) \cap [\rho_{j-1}(t), \rho_j(t)]\}$$

and $A_j(t) = (\lambda_j(t), \rho_j(t))$. Let $T \in [0,1]$ be defined by

$$T = \max\{\tau \colon h^*_{t,x} \in [f^*_{x_j}] \text{ for } x \in A_j(t) \text{ and all } t \leq \tau\}.$$

Suppose that $T < 1$. Choose $x_j \in A_j(T)$ such that $\bar{h}_T(x_j) = 0$. There exists $\epsilon_j > 0$ such that $|\bar{h}_t(x_j)| < 1$ for all $|t - T| < \epsilon_j$. Therefore $h_t(p^{-1}(x_j)) \subseteq p^{-1}((-1,1)) = Y \times (-1,1)$ and so $h^*_{t,x_j} \in [f^*_{x_j}]$ for all $t < T + \epsilon_j$ because h^*_{t,x_j} and h^*_{T,x_j} are homotopic. Thus a fiber homotopy h_t in which $\bar{h}_t$ deleted $[\rho_j(t), \lambda_{j+1}(t)]$ for $t < T + \epsilon$, where $\epsilon = \min\{\epsilon_j, \epsilon_{j+1}\}$, would induce a homotopy between h^*_{t,x_j}

and $h^*_{t,x_{j+1}}$. This would imply that $[f^*_{x_j}] = [f^*_{x_{j+1}}]$, contrary to the definition of the reduced type $\tau_0(f)$. By the same reasoning, since $[f^*_{x_{j-1}}] \neq [f^*_{x_j}]$ in $\tau_0(f)$, there cannot be a liftable homotopy $\bar{h}_t$ that deletes $[\rho_{j-1}(T), \lambda_j(T)]$. Therefore $h_{t,x} \in [f^*_{x_j}]$ for some $t > T$, contrary to the definition of T. We conclude that $T = 1$, that is, $[f^*_{x_j}] \in \tau(g)$. We note that it is possible to fiber homotope f so that some $\tau(h_t)$ contains $(\dots, [f^*_{x_j}], [f^*_{x_j}], [f^*_{x_j}], \dots)$ as we did in defining the map k_1 of Section 2. However, removing two copies of $[f^*_{x_j}]$ by a fiber homotopy as in the proof of Theorem 4.1 still leaves $[f^*_{x_j}] \in \tau(g)$.

By Lemma 5.1, if $\tau(g) = ([g^*_{x_1}], \dots, [g^*_{x_m}])$, then g has at least

$$\sum_{k=1}^{m} MF_0[g^*_{x_k}] \geq \sum_{j=1}^{m_0} MF_0[f^*_{x_j}]$$

fixed points and therefore

$$MF(f; \mathfrak{F}_{\mathcal{S}}) \geq \sum_{j=1}^{m_0} MF_0[f^*_{x_j}].$$

Theorem 5.1 proves that this inequality is an equality. □

6 Examples

It is a classical result that if $g\colon S^1 \to S^1$ is of degree $\deg(g)$, then $MF(g) = |\deg(g) - 1|$. Thus we have

Proposition 6.1. *If $f\colon \mathfrak{S}^2 \to \mathfrak{S}^2$ is a fiber map of reduced type $\tau_0(f) = ([f^*_{x_1}], \dots, [f^*_{x_m}])$, then $MF(f; \mathfrak{S}^2) = \sum_{j=1}^{m} MF_0[f^*_{x_j}]$ where, if $MF_0[f^*_{x_j}] \neq 1$, then $MF_0[f^*_{x_j}] = |\deg(f^*_{x_j}) - 1|$.*

The map f of Section 2 is a reduced skeletal fiber map of type $\tau(f) = (a, b)$ with $\bar{f}(-1) = -1$ and a fixed point of $\bar{f}$ at $x = -\frac{1}{3}$ such that the degree of $f_{-\frac{1}{3}}$ is a, so $MF(f; \mathfrak{S}^2) = |a - 1| + 1$ as we claimed.

Since every map $g\colon S^r \to S^r$ a map of degree $deg(g) \neq (-1)^{r-1}$ has a fixed point, but it is homotopic to a map with a single fixed point, and a map of degree $(-1)^{r-1}$ is homotopic to the antipodal map, we have the following result.

Proposition 6.2. *If $f\colon \mathfrak{S}^n \to \mathfrak{S}^n, n \geq 3$ is a fiber map of reduced type $\tau_0(f) = ([f^*_{x_1}], \dots, [f^*_{x_m}])$, then $MF(f; \mathfrak{S}^n) = \sum_{j=1}^{m} MF_0[f^*_{x_j}]$ where $MF_0[f^*_{x_j}] = 0$ if $\deg(f^*_{x_j}) = (-1)^n$ and $MF_0[f^*_{x_j}] = 1$ otherwise* *except that* $\bar{f}(-1) = -1$ *implies* $MF_0[f^*_{x_1}] = 1$ *and* $\bar{f}(1) = 1$ *implies* $MF_0[f^*_{x_m}] = 1$.

As we remarked in the introduction, by Theorem 8.2 of [7] if f is a fiber map f of a fiber space $\mathfrak{F}$ of the form $p\colon E \to X$ with fiber Y in which E, X and Y are all simply-connected then $MF(f; \mathfrak{F}) \leq 1$. In contrast, although for the MS fibering $\mathfrak{S}^3$ we have $E = S^3, X = [-1,1]$ and $Y = S^2$ yet

Corollary 6.1. *Given a natural number k, there is a fiber map $f^{[k]}\colon \mathfrak{S}^3 \to \mathfrak{S}^3$ such that $MF(f^{[k]}; \mathfrak{S}^3) = k$.*

Proof. We will construct a skeletal fiber map $f^{[k]}$, first defining a skeletal map $\bar{f}^{[k]}\colon [-1,1] \to [-1,1]$. The skeletal vertices of $\bar{f}^{[k]}$ are $\rho_j = \frac{2j-k}{k}$ for $j = 0, 1, \ldots, k$. Define $\bar{f}^{[k]}\colon (\rho_{j-1}, \rho_j) = A_j \to (-1,1)$ for each j by

$$\bar{f}^{[k]}(x) = (-1)^j(kx + k - 2j + 1)$$

and extend to $[-1,1]$ by letting $\bar{f}^{[k]}(\rho_j) = (-1)^j$. For $j = 1, \ldots, k$, let $f^*_{x_j}\colon S^2 \to S^2$ be a constant map if j is even and the identity map if j is odd. Then we obtain $f^{[k]}\colon S^3 \to S^3$ by setting

$$f^{[k]}(e) = f^{[k]}(x,y) = (\bar{f}^{[k]}(x), f^*_{x_j}(y))$$

if $e = (x,y) \in p^{-1}(A_j)$ for some j and, if $e \in p^{-1}(\rho_j)$ then $f^{[k]}(e) = \sigma$ if j is even and $f^{[k]}(e) = \nu$ if j is odd. The type $\tau(f^{[k]}) = ([f^*_{x_1}], \ldots, [f^*_{x_k}])$ of $f^{[k]}$ is reduced. Moreover, $MF[f^*_{x_j}] = 1$ by Proposition 6.2 because no f^*_j is of degree -1 and, if k is even, $\bar{f}^{[k]}(1) = 1$. Therefore,

$$MF(f^{[k]}; \mathfrak{S}^3) = \sum_{j=1}^{k} MF_0[f^*_{x_j}] = k.$$

□

We conclude with an example in which E is not a sphere. Let T^r be the r-torus and consider $\mathfrak{F}_{\mathcal{S}} = \Sigma(p_0\colon T^r \to \{0\})$. For a fiber map $f\colon \mathfrak{F}_{\mathcal{S}} \to \mathfrak{F}_{\mathcal{S}}$ with reduced type $\tau_0(f) = ([f^*_{x_1}], \ldots, [f^*_{x_m}])$ let $F^*_{x_j}$ be the r-by-r integer matrix of the homomorphism of the fundamental group induced by $f^*_{x_j}$ and let I_r denote the r-by-r identity matrix. Then the Nielsen number of $f^*_{x_j}$ is $N(f^*_{x_j}) = |\det(I_r - F^*_{x_j})|$ ([9], page 33) and $MF(f^*_{x_j}) = N(f^*_{x_j})$ by [10]. Thus, by Theorem 5.2, $MF(f; \mathfrak{F}_{\mathcal{S}}) = \sum_{j=1}^{m} MF_0[f^*_{x_j}]$ where $MF_0[f^*_{x_j}] = |\det(I_r - F^*_{x_j})|$ except that $\bar{f}(-1) = -1$ implies $MF_0[f^*_{x_1}] = 1$ and $\bar{f}(1) = 1$ implies $MF_0[f^*_{x_m}] = 1$.

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