COINCIDENCES OF PROJECTIONS AND LINEAR n-VALUED MAPS OF TORI

Robert F. Brown
Department of Mathematics
University of California
Los Angeles, CA 90095
email: rfb@math.ucla.edu

Jon T. Lo Kim Lin*
Department of Mathematics
University of California
Berkeley, CA 94720
email: jonlokimlin@berkeley.edu

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Abstract

We prove that the Nielsen fixed point number $N(\varphi)$ of an n-valued map $\varphi\colon X \multimap X$ of a compact connected triangulated orientable q-manifold without boundary is equal to the Nielsen coincidence number of the projections of the graph of φ, a subset of $X \times X$, to the two factors. For certain $q \times q$ integer matrices A, there exist "linear" n-valued maps $\Phi_{n,A,\sigma}\colon T^q \multimap T^q$ of q-tori that generalize the single-valued maps $f_A\colon T^q \to T^q$ induced by the linear transformations $T_A\colon \mathbb{R}^q \to \mathbb{R}^q$ defined by $T_A(v) = Av$. By calculating the Nielsen coincidence number of the projections of its graph, we calculate $N(\Phi_{n,A,\sigma})$ for a large class of linear n-valued maps.

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1 Introduction

An n-valued map $\varphi\colon X \multimap Y$ is a continuous, that is both upper and lower semicontinous, multivalued function such that $\varphi(x)$ is an unordered subset of exactly n points of Y. For X a finite polyhedron and $\varphi\colon X \multimap X$ an n-valued map, the Nielsen number of φ was introduced by Schirmer in [6],[1] but Schirmer did not discuss the

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[1] Schirmer called such multivalued functions "n-valued continuous multifunctions" and that was shortened to "n-valued multimap" in [1] and subsequent papers. However, a function

calculation of the Nielsen number. The Nielsen numbers of n-valued maps of the circle were calculated in [1] and of n-valued fiber maps of the total space of a fibration over the circle in [2]. In this paper, we extend further the class of n-valued maps for which the Nielsen number can be calculated.

The *graph* $\Gamma(\varphi)$ of a multivalued map $\varphi\colon X \multimap X$ is defined by

$$\Gamma(\varphi) = \{(x, y) \in X \times X \colon y \in \varphi(x)\}.$$

The genesis of our paper was a comment that Andrew Cotton-Clay made to one of the authors. He pointed out that a fixed point of a multivalued map can be viewed as a coincidence of two single-valued functions, as follows: let $\Pi_1^\varphi, \Pi_2^\varphi \colon \Gamma(\varphi) \to X$ be the projections defined by $\Pi_1^\varphi(x, y) = x$ and $\Pi_2^\varphi(x, y) = y$, then $x \in \varphi(x)$ if and only if Π_1^φ and Π_2^φ have a coincidence at (x, x). We will prove in Section 2 that, if X is a compact connected triangulated orientable manifold without boundary, this correspondence extends to a correspondence between the Nielsen theory of n-valued maps $\varphi\colon X \multimap X$ and the Nielsen coincidence theory of the projections $\Pi_1^\varphi, \Pi_2^\varphi \colon \Gamma(\varphi) \to X$. Consequently, in this setting, techniques for calculating Nielsen coincidence numbers can be used to compute Nielsen numbers of n-valued maps. In Section 3 we introduce a class of n-valued maps of tori. Given any q-by-q integer matrix, the linear transformation $T_A\colon \mathbb{R}^q \to \mathbb{R}^q$ defined by $T_A(v) = Av$ induces a “linear” map $f_A\colon T^q \to T^q$ of the q-torus. We show in Section 3 that, for many such matrices A, there is an n-valued map $\Phi_{n,A,\sigma}\colon T^q \multimap T^q$ based on A whose construction generalizes f_A; it will therefore be called “linear” also. To illustrate the calculation of the Nielsen number of an n-valued map as the Nielsen coincidence number of the projections of its graph, in Section 4 we calculate the Nielsen numbers of a large class of linear n-valued maps of tori.

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2 The Coincidence Nielsen Number of Projections

An n-valued map $\varphi\colon X \multimap Y$ is *split* if there exist single-valued maps $f_1, \dots f_n\colon X \to Y$ such that $\varphi(x) = \{f_1(x), \dots, f_n(x)\}$ for all $x \in X$.

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that is n-valued is evidently multivalued, so the present terminology of “n-valued map” seems adequate to describe a continuous function of this type.

The fundamental Splitting Lemma for n-valued maps states that, if $\varphi\colon X \multimap Y$ is an n-valued map then $\Pi_1^\varphi\colon \Gamma(\varphi) \to X$ is a covering space and therefore, if X is simply-connected, then φ is split.

Let $\varphi\colon X \multimap X$ be an n-valued map of a connected finite polyhedron. In [6], Schirmer defined $x_0, x_1 \in Fix(\varphi)$ to be equivalent if there is a path c in X from x_0 to x_1 such that, for $\{f_1, \ldots, f_n\}$ the splitting of $\varphi c\colon I \multimap X$, there is a j such that $f_j(0) = x_0, f_j(1) = x_1$ and the path f_j is homotopic to c relative to the endpoints. An equivalence class of fixed points of φ is a *fixed point class* of φ.

Let $\varphi, \psi\colon X \multimap X$ be n-valued maps and let d be the metric of X. Define the distance $d(\varphi, \psi)$ between φ and ψ to be the supremum of $\rho_d(\varphi(x), \psi(x))$ for all $x \in X$, where ρ_d denotes the Hausdorff metric induced by d.

Theorem 2.1. *([5], Theorem 6) Let $\varphi\colon X \multimap X$ be an n-valued map of a connected finite polyhedron. Given $\epsilon > 0$, there exists an n-valued map $\psi\colon X \multimap X$ such that ψ has finitely many fixed points, all in maximal simplices of X, and $d(\varphi, \psi) < \epsilon$.*

The fixed point index of φ at an isolated fixed point x, denoted $ind(\varphi, x)$, is defined in [6], page 210 in terms of the classical fixed point index by $ind(\varphi, x) = ind(f_j, x)$ where $\{f_1, \ldots, f_n\}$ is the splitting of φ in a neighborhood of x and $f_j(x) = x$. Let U be a neighborhood of a fixed point class $\mathbf{F}$ whose closure contains no other fixed points of φ, then, for a suitably chosen ϵ, Theorem 2.1 approximates the restriction of φ by a fix-finite n-valued map ψ and the index $ind(\mathbf{F})$ of the fixed point class $\mathbf{F}$ is defined to be the sum of the indices of the fixed points of ψ in U. The *Nielsen number* $N(\varphi)$ is the number of *essential* fixed point classes, that is, those of nonzero index.

The *coincidences* of the projections $\Pi_1^\varphi, \Pi_2^\varphi\colon \Gamma(\varphi) \to X$ are the points (x, x) in $\Gamma(\varphi)$. We will say that coincidence points (x_0, x_0) and (x_1, x_1) are *C-equivalent* if they are in the same Nielsen coincidence class, that is, if there is a path $\eta\colon I \to \Gamma(\varphi)$ such that $\eta(0) = (x_0, x_0), \eta(1) = (x_1, x_1)$ and the paths $\Pi_1^\varphi\eta, \Pi_2^\varphi\eta\colon I \to X$ are homotopic in X relative to the endpoints.

Lemma 2.1. *Fixed points x_0, x_1 of an n-valued map $\varphi\colon X \multimap X$ are equivalent if and only if (x_0, x_0) and (x_1, x_1) are C-equivalent.*

Proof. Suppose x_0 and x_1 are equivalent fixed points of φ. Then there exists $c\colon I \to X$ such that $c(0) = x_0, c(1) = x_1$ and, for $\varphi c = \{f_1, \ldots, f_n\}\colon I \multimap X$ there exists f_j such that $f_j(0) = x_0, f_j(1) = x_1$ and f_j and c are homotopic relative to the endpoints. Then (x_0, x_0)

and (x_1, x_1) are C-equivalent because $\eta\colon I \to \Gamma(\varphi)$ defined by $\eta(t) = (c(t), f_j(t))$ is the required path.

Now let (x_0, x_0) and (x_1, x_1) be C-equivalent by $\eta\colon I \to \Gamma(\varphi)$, that is, $\eta(0) = (x_0, x_0), \eta(1) = (x_1, x_1)$ and $\Pi_1^\varphi \eta$ is homotopic to $\Pi_2^\varphi \eta$ in X relative to the endpoints. Let $c = \Pi_1^\varphi \eta\colon I \to X$, then η is a lift of c to the covering space $\Pi_1^\varphi\colon \Gamma(\varphi) \to X$. Now $\varphi c = \{f_1, \ldots, f_n\}$ so

$$\varphi c(0) = \varphi \Pi_1^\varphi \eta(0) = \varphi(x_0) = \{f_1(0), \ldots, f_n(0)\}$$

and $f_j(0) = x_0$ for some j since $x_0 \in \varphi(x_0)$. Define $\delta\colon I \to \Gamma(\varphi)$ by $\delta(t) = (\Pi_1^\varphi \eta(t), f_j(t))$, which is well-defined because $f_j(t) \in \varphi \Pi_1^\varphi \eta(t)$. Since δ is a lift of c and $\delta(0) = (x_0, x_0)$, then $\delta = \eta$ and therefore $f_j = \Pi_2^\varphi \eta$ so $c = \Pi_1^\varphi \eta$ is homotopic to f_j and thus x_0 and x_1 are equivalent. $\square$

For the rest of the paper, X will denote a compact connnected triangulated orientable q-manifold without boundary.

Lemma 2.2. *Let x_0 be an isolated fixed point of an n-valued map $\psi\colon X \multimap X$, then $ind(\psi, x_0)$ equals the index of (x_0, x_0) as a coincidence point of $\Pi_1^\psi, \Pi_2^\psi\colon \Gamma(\psi) \to X$.*

Proof. Let x_0 be a fixed point of ψ and let U be a euclidean neighborhood of x_0 that contains no other fixed point of ψ. Restricting ψ to U, we obtain the split n-valued map $\psi|U = \{f_1, f_2, \ldots, f_n\}\colon U \multimap X$ such that $f_j(x_0) = x_0$ for some j. The component U_j of $(\Pi_1^\psi)^{-1}(U) \subseteq \Gamma(\psi)$ containing (x_0, x_0) is homeomorphic to U and consists of the points $(x, f_j(x))$ for $x \in U$. Identifying U and U_j with euclidean q-space so that x_0 and (x_0, x_0) correspond to the origin, let $S \subseteq U$ correspond to an $(q-1)$-sphere of radius small enough so that $f_j(S) \subseteq U$. Let $S_j = (\Pi_1^\psi)^{-1}(S) \cap U_j$, then $\Pi_1^\psi - \Pi_2^\psi\colon S_j \to U - \{x_0\}$ induces $(\Pi_1^\psi - \Pi_2^\psi)_*\colon H_{q-1}(S_j) \to H_{q-1}(U - \{x_0\})$ which, with appropriate choices of generators, determines the index of (x_0, x_0) as a coincidence point of Π_1^ψ and Π_2^ψ. Noting that, for $(x, f_j(x)) \in S_j$, we have $\Pi_1^\psi(x, f_j(x)) = x$ and $\Pi_2^\psi(x, f_j(x)) = f_j(x)$, we may identify $(\Pi_1^\psi - \Pi_2^\psi)_*$ with $(id - f_j)_*\colon H_{q-1}(S) \to H_{q-1}(U - \{x_0\})$ which determines the fixed point index $ind(\psi, x_0)$. $\square$

Let $\varphi\colon X \multimap X$ be an n-valued map and $x \in X$ such that $\varphi(x) = \{y_1, y_2, \ldots, y_n\}$ and let $\gamma(x, \varphi)$ denote the minimum distance among the points y_i. Then $\gamma(\varphi)$, the *gap* of φ, is the infimum of the $\gamma(x, \varphi)$ for all $x \in X$. Schirmer observed in [5] that the continuity of φ and the compactness of X imply that $\gamma(\varphi) > 0$.

An *ϵ-homotopy* of n-valued maps is an n-valued map $\Phi = \{\varphi_t\}\colon X \times I \multimap X$, where $\varphi_t(x) = \Phi(x, t)$, such that $d(\varphi_s, \varphi_t) < \epsilon$ for all

$s, t \in I$. If $\varphi, \psi\colon X \multimap X$ are n-valued maps such that $\varphi = \varphi_0$ and $\psi = \varphi_1$ for some ϵ-homotopy $\Phi = \{\varphi_t\}$, then φ and ψ are said to be *ϵ-homotopic*.

For $\varphi\colon X \multimap X$ an n-valued map of X, we denote by $N(\Pi_1^\varphi, \Pi_2^\varphi)$ the Nielsen coincidence number of the projections $\Pi_1^\varphi, \Pi_2^\varphi\colon \Gamma(\varphi) \to X$.

Lemma 2.3. *If $\varphi, \psi\colon X \multimap X$ are ϵ-homotopic n-valued maps, where $\epsilon < \frac{\gamma(\varphi)}{3}$, then $N(\Pi_1^\varphi, \Pi_2^\varphi) = N(\Pi_1^\psi, \Pi_2^\psi)$.*

Proof. For $x \in X$, write $\varphi(x) = \{y_1, \dots, y_n\}$ and $\psi(x) = \{z_1, \dots, z_n\}$. For $y_i \in \varphi(x)$, we may number the z_j so that $d(y_i, z_i) < \epsilon$ and $d(y_i, z_j) > \epsilon$ for $j \neq i$ because

$$\rho_d(\varphi(x), \psi(x)) < \epsilon < \tfrac{\gamma(\varphi)}{3}.$$

Define $h\colon \Gamma(\varphi) \to \Gamma(\psi)$ by $h(x, y_i) = (x, z_i)$, which is one-to-one and onto. Since $\Pi_1^\varphi\colon \Gamma(\varphi) \to X$ and $\Pi_1^\psi\colon \Gamma(\psi) \to X$ are covering spaces, there is a neighborhood U of x such that the restrictions of Π_1^φ to $(\Pi_1^\varphi)^{-1}(U)$ and Π_1^ψ to $(\Pi_1^\psi)^{-1}(U)$ are local homeomorphisms. Let U_i be the component of $(\Pi_1^\varphi)^{-1}(U)$ that contains (x, y_i) and V_i the component of $(\Pi_1^\psi)^{-1}(U)$ that contains (x, z_i). Then h agrees with the homeomorphism $(\Pi_1^\psi)^{-1}\Pi_1^\varphi\colon U_i \to V_i$ on a neighborhood of (x, y_i) so it is continuous and therefore a homeomorphism.

Let $\Delta\colon X \times I \multimap X$ be an n-valued ϵ-homotopy such that $\Delta(x, 0) = \varphi(x)$ and $\Delta(x, 1) = \psi(x)$ for all $x \in X$. Define a single-valued homotopy $H\colon \Gamma(\varphi) \times I \to X$ by

$$H((x, y), t) = \Delta(x, t) \cap B_\epsilon(y)$$

where

$$B_\epsilon(y) = \{y' \in X\colon d(y, y') < \epsilon\}.$$

The function H is well-defined because $\epsilon < \gamma(\varphi)/3$ implies that only one of the n points of $\Delta(x, t)$ can lie in $B_\epsilon(y)$.

We will prove that H is continuous by showing that, given a point $((x^*, y^*), t^*) \in \Gamma(\varphi) \times I$ and an open subset U of X containing $H((x^*, y^*), t^*)$, there is an open subset V of $\Gamma(\varphi) \times I$ containing $((x^*, y^*), t^*)$ such that $H(V) \subseteq U$. Let $W = B_\epsilon(y^*) \cap U$ then, since Δ is lower semi-continuous, the set

$$\Delta_+^{-1}(W) = \{(x, t) \in X \times I\colon \Delta(x, t) \cap W \neq \emptyset\}$$

is open in $X \times I$. Consider the map $\Pi_1^\varphi \times id\colon \Gamma(\varphi) \times I \to X \times I$ and let V be an η-neighborhood of $((x^*, y^*), t^*)$ in $(\Pi_1^\varphi \times id)^{-1}(\Delta_+^{-1}(W))$ for some $\eta \leq \epsilon$. Suppose $((x', y'), t') \in V$, since

$$(\Pi_1^\varphi \times id)((x', y'), t') = (x', t') \in \Delta_+^{-1}(W)$$

we have

$$\Delta(x', t') \cap W = \Delta(x', t') \cap (B_\epsilon(y^*) \cap U) \neq \emptyset.$$

Writing $\Delta(x', t') = \{y_1(t'), \ldots, y_n(t')\}$, then $\Delta(x', t') \cap W = y_i(t')$ for some i because no other $y_j(t')$ is in $W \subset B_\epsilon(y^*)$ for the following reason. If it were, then $d(y_i(t'), y_j(t')) < \epsilon$. But Δ is an ϵ-homotopy so $d(y_i(0), y_i(t')) < \epsilon$ and $d(y_j(0), y_j(t')) < \epsilon$ for some $y_i(0), y_j(0) \in \varphi(x')$ and this would imply that $d(y_i(0), y_j(0)) < 3\epsilon$ whereas $3\epsilon < \gamma(\varphi)$. The definition of V implies that $d(y', y^*) < \eta \leq \epsilon$ so if $y_i(t') \in B_\epsilon(y^*)$ then $d(y', y_i(t')) < 2\epsilon$ and it must be that $y_i(0) = y'$, that is

$$y_i(t') = \Delta(x', t') \cap B_\epsilon(y') = H((x', y'), t').$$

Since $y_i(t') \in W \subseteq U$, we have proved that $H(x', y'), t') \in U$ and thus $H(V) \subseteq U$ which establishes the continuity of $H\colon \Gamma_\varphi \times I \to X$.

Now

$$H((x, y), 1) = \Delta(x, 1) \cap B_\epsilon(y) = \psi(x) \cap B_\epsilon(y) = z$$

where z is the member of the set $\psi(x)$ in $B_\epsilon(y)$. By definition, $h(x, y) = (x, z)$ so

$$H((x, y), 1) = z = \Pi_2^\psi h(x, y).$$

On the other hand,

$$H((x, y), 0) = \Delta(x, 0) \cap B_\epsilon(y) = \varphi(x) \cap B_\epsilon(y) = y = \Pi_2^\varphi(x, y).$$

Therefore $\Pi_2^\varphi, \Pi_2^\psi h\colon \Gamma(\varphi) \to X$ are homotopic and since $\Pi_1^\varphi = \Pi_1^\psi h$, the homotopy invariance of the Nielsen coincidence number implies that $N(\Pi_1^\varphi, \Pi_2^\varphi)$ is equal to the Nielsen coincidence number of $h\Pi_1^\psi, h\Pi_2^\psi\colon \Gamma(\varphi) \to X$. Because h is a homeomorphism, $N(h\Pi_1^\psi, h\Pi_2^\psi) = N(\Pi_1^\psi, \Pi_2^\psi)$ and we have proved that $N(\Pi_1^\varphi, \Pi_2^\varphi) = N(\Pi_1^\psi, \Pi_2^\psi)$. $\square$

Theorem 2.2. *If $\varphi\colon X \multimap X$ is an n-valued map of a compact connected triangulated orientable manifold without boundary, then $N(\varphi) = N(\Pi_1^\varphi, \Pi_2^\varphi)$.*

Proof. By Lemma 4.1 of [6], given $\epsilon < \frac{\gamma(\varphi)}{3}$ there exists $\delta \leq \epsilon$ such that if $d(\varphi, \psi) < \delta$ then ψ is ϵ-homotopic to φ. By Theorem 2.1, there is an n-valued map ψ such that ψ has finitely many fixed points and $d(\varphi, \psi) < \delta$. So ψ and φ are ϵ-homotopic and therefore $N(\Pi_1^\varphi, \Pi_2^\varphi) = N(\Pi_1^\psi, \Pi_2^\psi)$ by Lemma 2.3.

By Lemma 2.1, letting each of the finite number of fixed points of ψ correspond to $(x_i, x_i) \in \Gamma(\psi)$ defines a one-to-one correspondence between the fixed point classes of ψ and the coincidence classes of the projections $\Pi_1^\psi, \Pi_2^\psi \colon \Gamma(\psi) \to X$. Let $\mathbf{F}$ be a fixed point class of ψ and $\mathbf{C}$ the corresponding coincidence class then, by Lemma 2.2, $ind(\mathbf{F}) = ind(\mathbf{C})$ and therefore $N(\psi) = N(\Pi_1^\psi, \Pi_2^\psi)$. Since $N(\varphi) = N(\psi)$ by Theorem 6.5 of [6], we conclude that $N(\varphi) = N(\Pi_1^\varphi, \Pi_2^\varphi)$. $\square$

3 Linear n-valued maps of tori

Let $P_q \colon \mathbb{R}^q \to T^q = S^1 \times \cdots S^1$ be the universal covering space, that is, for $t = (t_0, \ldots, t_{q-1}) \in \mathbb{R}^q$, let $P_q(t) = (p(t_0), \ldots, p(t_{q-1}))$ where $p(t_j) = e^{i2\pi t_j}$. For $q \geq 1$, define $\mathbb{Z}_q = \{0, 1, \ldots, q-1\}$. Let $A = (a_{ij})_{i,j \in \mathbb{Z}_q}$ be a $q \times q$ integer matrix. The linear transformation $T_A \colon \mathbb{R}^q \to \mathbb{R}^q$ is defined by

$$T_A(t) = (A_0 t, \ldots, A_{q-1} t)$$

where A_i denotes the i-th row of A and $A_i t$ is the dot product. The map $f_A \colon T^q \to T^q$ defined by

$$f_A(p(t)) = (p(A_0 t), \ldots, p(A_{q-1} t))$$

is called a *linear map* of the q-torus because the lift of f_A to the universal covering space that maps the origin to itself is the linear transformation T_A.

Given an integer $n \geq 1$, a $q \times q$ matrix A and $\sigma = (\sigma_0, \ldots, \sigma_{q-1}) \in (\mathbb{Z}_2)^q$, a nonzero vector where each $\sigma_j = 0$ or 1, we define an n-valued map $\widetilde{\Phi}_{n,A,\sigma} \colon I^q = I \times \cdots \times I \multimap T^q$ by

$$\widetilde{\Phi}_{n,A,\sigma}(t) = \{\widetilde{\Phi}_{n,A,\sigma}^{(0)}(t), \ldots, \widetilde{\Phi}_{n,A,\sigma}^{(n-1)}(t)\}$$

where, for $k \in \mathbb{Z}_n$, we set

$$\widetilde{\Phi}_{n,A,\sigma}^{(k)}(t) = P_q(\tfrac{1}{n}(At + k\sigma)).$$

For $t = (t_0, \ldots, t_{q-1}) \in \partial I^q$ such that $t_{i^*} = 0$, write t as $t(i_0^*)$, that is $t(i_0^*) = (t_0, \ldots, t_{i^*} = 0, \ldots, t_{q-1})$, and let $t(i_1^*) = (t_0, \ldots, t_{i^*} = 1, \ldots, t_{q-1})$. Thus the coordinates of $t(i_1^*)$ are the same as those of $t(i_0^*)$ except that t_{i^*} is changed from 0 to 1. Then $\widetilde{\Phi}_{n,A,\sigma} : I^q \multimap T^q$ induces a linear n-valued map $\Phi_{n,A,\sigma} \colon T^q \multimap T^q$ if and only if

$$\widetilde{\Phi}_{n,A,\sigma}(t(i_0^*)) = \widetilde{\Phi}_{n,A,\sigma}(t(i_1^*)),$$

as unordered sets of n points, for all such pairs $t(i_0^*), t(i_1^*) \in \partial I^q$.

If $n = 1$, then

$$\widetilde{\Phi}_{n,A,\sigma}(t) = \widetilde{\Phi}^{(0)}_{n,A,\sigma}(t) = P_q(At) = P_q T_A(t)$$

for any σ so $\widetilde{\Phi}_{n,A,\sigma}$ induces the linear map $f_A \colon T^q \to T^q$. Since the $\Phi_{n,A,\sigma} \colon T^q \multimap T^q$ may thus be viewed as n-valued generalizations of such linear maps, we will call them *linear n-valued maps of T^q*.

If $q = 1$, then $\sigma = (1)$ and, writing the 1×1 matrix $A = [d]$, we have

$$\begin{aligned} \widetilde{\Phi}_{n,A,\sigma}(t) &= \{\widetilde{\Phi}^{(0)}_{n,A,\sigma}(t), \dots, \widetilde{\Phi}^{(n-1)}_{n,A,\sigma}(t)\} \\ &= \{p(\tfrac{1}{n}dt), \dots, p(\tfrac{1}{n}(dt + (n-1))\} \end{aligned}$$

and thus the induced map $\Phi_{n,A,\sigma} \colon S^1 \multimap S^1$ is the n-valued power map $\phi_{n,d}$ of [1].

Although every $q \times q$ integer matrix induces a single-valued linear map of T^q, there are strong restrictions on the matrix A in order for it to induce a linear n-valued map $\Phi_{n,A,\sigma} \colon T^q \multimap T^q$ for $n > 1$.

Define $x = (x_0, \dots, x_{q-1}), y = (y_0, \dots, y_{q-1}) \in \mathbb{Z}^q$ to be *congruent mod n*, written $x \equiv y$ mod n, if $x_j \equiv y_j$ mod n for all $j \in \mathbb{Z}_q$.

Theorem 3.1. *A $q \times q$ integer matrix A and nonzero $\sigma \in (\mathbb{Z}_2)^q$ induce a linear n-valued map $\Phi_{n,A,\sigma} \colon T^q \multimap T^q$ if and only if there exists $\zeta(A) = (\zeta_0(A), \dots, \zeta_{q-1}(A)) \in \mathbb{Z}^q$ such that $A_j \equiv \sigma_j \zeta(A)$ mod n for all $j \in \mathbb{Z}_q$.*

Proof. Suppose $t(i_0^*), t(i_1^*) \in \partial I^q$, then

$$\widetilde{\Phi}_{n,A,\sigma}(t(i_0^*)) = \widetilde{\Phi}_{n,A,\sigma}(t(i_1^*))$$

as unordered sets of n points if and only if for each $k \in \mathbb{Z}_n$ there exists a unique $\ell \in \mathbb{Z}_n$ such that

$$\widetilde{\Phi}^{(k)}_{n,A,\sigma}(t(i_0^*)) = \widetilde{\Phi}^{(\ell)}_{n,A,\sigma}(t(i_1^*)).$$

We calculate

$$\begin{aligned} \widetilde{\Phi}^{(k)}_{n,A,\sigma}(t(i_0^*) = \Bigg(& p(\tfrac{1}{n}(\sum_{\substack{j=0 \\ j \neq i^*}}^{q-1} a_{0j} t_j + k\sigma_0), \dots \\ & \dots, p(\tfrac{1}{n}(\sum_{\substack{j=0 \\ j \neq i^*}}^{q-1} a_{q-1,j} t_j + k\sigma_{q-1})) \Bigg) \end{aligned}$$

and

$$\widetilde{\Phi}^{(\ell)}_{n,A,\sigma}(t(i_1^*)) = \Big(p(\tfrac{1}{n}(a_{0i^*} + \sum_{\substack{j=0\\ j\neq i^*}}^{q-1} a_{0j}t_j + \ell\sigma_0)), \ldots$$

$$\ldots, p(\tfrac{1}{n}(a_{q-1,i^*} + \sum_{\substack{j=0\\ j\neq i^*}}^{q-1} a_{q-1,j}t_j + \ell\sigma_{q-1})) \Big)$$

Therefore, we have equality if and only if

$$k\sigma_u \equiv a_{ui^*} + \ell\sigma_u \mod n$$

for all $u \in \mathbb{Z}_q$. If $\sigma_u = 0$ the congruence becomes $a_{ui^*} \equiv 0 \mod n$ for each $i^* \in \mathbb{Z}_q$. If $\sigma_u = 1$ then we have

$$k \equiv a_{ui^*} + \ell \mod n$$

and setting $\zeta_{i^*}(A) = k - \ell$ yields the desired result. □

Proposition 3.1. *A linear n-valued map $\Phi_{n,A,\sigma}\colon T^q \multimap T^q$ is split if and only if all a_{ij} in A are divisible by n.*

Proof. The n-valued map $\Phi_{n,A,\sigma}$ splits if and only if

$$\widetilde{\Phi}^{(k)}_{n,A,\sigma}(t(i_0^*)) = \widetilde{\Phi}^{(k)}_{n,A,\sigma}(t(i_1^*))$$

for each k and all pairs $t(i_0^*), t(i_1^*) \in \partial I^q$ so that $\Phi_{n,A,\sigma} = (f_0, \ldots, f_{q-1})$ where f_k is induced by $\widetilde{\Phi}^{(k)}_{n,A,\sigma}$. Therefore $\zeta_j(A) = k - k = 0$ for all $j \in \mathbb{Z}_q$ and we conclude that all the a_{ij} are divisible by n. □

Example 3.1. *There is a linear 3-valued map $\Phi_{3,A,\sigma}\colon T^4 \multimap T^4$ for $\sigma = (1,0,1,1)$ and*

$$A = \begin{bmatrix} -2 & 5 & 3 & 1 \\ 3 & 0 & 6 & -3 \\ 1 & -1 & 0 & 4 \\ 4 & 2 & -3 & 1 \end{bmatrix}$$

because the condition of Theorem 3.1 is satisfied for $\zeta(A) = (1,2,0,1)$.

4 Nielsen numbers of the linear n-valued maps

To compute the Nielsen numbers of linear n-valued maps of tori, we will apply the following result.

Theorem 4.1. *([3], Lemma 7.3) Let* $f, g\colon T^q \to T^q$ *be maps and let* F, G *be* $q \times q$ *integer matrices representing* $f_*, g_*\colon \pi_1(T^q) \to \pi_1(T^q)$, *then the Nielsen coincidence number* $N(f,g) = |\det(F-G)|$.

We will first illustrate how Theorem 2.2 is used by computing the Nielsen numbers of the n-valued power maps $\phi_{n,d}\colon S^1 \multimap S^1$ as a consequence of Theorem 4.1. This argument will then be generalized in order to calculate the Nielsen numbers of many of the linear n-valued maps $\Phi_{n,A,\sigma}\colon T^q \multimap T^q$. Assume first that n and d are relatively prime. The proof of Proposition 5.2 of [2] shows that the path

$$\begin{aligned} G_{n,d} &= \left\{(x, \frac{d}{n}x) \in \mathbb{R}^2\colon 0 \le x \le n\right\} \\ &= \{(nt, dt) \in \mathbb{R}^2\colon 0 \le t \le 1\} \end{aligned}$$

is a lift to the universal covering space $P_2\colon \mathbb{R}^2 \to T^2$ of the simple closed curve $\Gamma(\varphi_{n,d})$. The projections $\Pi_1', \Pi_2'\colon \mathbb{R}^2 \to \mathbb{R}$ of each factor are lifts of the projections $\Pi_1, \Pi_2\colon T^2 \to S^1$. Defining $g_{n,d}\colon I \to \mathbb{R}^2$ by $g_{n,d}(t) = (nt, dt)$ so that $g_{n,d}(I) = G_{n,d}$, we have $\Pi_1' g_{n,d}(t) = nt$ and $\Pi_2' g_{n,d}(t) = dt$. Therefore, for the induced fundamental group homomorphisms $\Pi_{1*}^{\varphi_{n,d}}, \Pi_{2*}^{\varphi_{n,d}}\colon \pi_1(\Gamma(\varphi_{n,d})) \to \pi_1(S^1)$, we see that $\Pi_{1*}^{\varphi_{n,d}}$ is multiplication by n and $\Pi_{2*}^{\varphi_{n,d}}$ is multiplication by d. It follows by Theorem 4.1 and Theorem 2.2 that $N(\varphi_{n,d}) = N(\Pi_1^{\varphi_{n,d}}, \Pi_2^{\varphi_{n,d}}) = |n-d|$. If n and d are not relatively prime, let w be their greatest common divisor. The proof of Proposition 5.2 of [2] also demonstrates that $\Gamma(\varphi_{n,d})$ has w components, each of which can be identified with $\Gamma(\varphi_{n/w,d/w})$. Since $\frac{n}{w}$ and $\frac{d}{w}$ are relatively prime, we have shown that $N(\varphi_{n/w,d/w}) = |\frac{n}{w} - \frac{d}{w}|$ so

$$N(\varphi_{n,d}) = w|\tfrac{n}{w} - \tfrac{d}{w}| = |n-d|$$

for all n and d. This result was obtained as Theorem 4.1 of [1] by an entirely different method.

We extend this calculation of the Nielsen number of the n-valued power maps of the circle to many of the linear n-valued maps of tori as follows.

Theorem 4.2. *If* $\Phi_{n,A,\sigma}\colon T^q \multimap T^q$ *is a linear n-valued map such that at least one* $\zeta_j(A)$ *is relatively prime to n, then* $\Gamma(\Phi_{n,A,\sigma})$ *is connected and*

$$N(\Phi_{n,A,\sigma}) = n|\det(E - \tfrac{1}{n}A)|$$

where E is the $q \times q$ *identity matrix.*

Proof. We may assume that $\zeta_0(A)$ and n are relatively prime. Let D_n be the diagonal matrix $D_n = \operatorname{diag}(n, 1, \ldots, 1)$. Define $g_{n,A,\sigma}\colon I^q \to \mathbb{R}^{2q}$ by

$$g_{n,A,\sigma}(t) = (D_n t, \tfrac{1}{n} A D_n t)$$

and set $G_{n,A,\sigma} = g_{n,A,\sigma}(I^q)$. We claim that $G_{n,A,\sigma}$ is a lift of $\Gamma(\Phi_{n,A,\sigma})$ to the covering space $P_{2q}\colon \mathbb{R}^{2q} \to T^{2q}$.

A point $x \in T^{2q}$ is in $\Gamma(\Phi_{n,A,\sigma})$ if it is of the form

$$x = (P_q(\tau), P_q(\tfrac{1}{n}(A\tau + k\sigma)))$$

for some $k \in \mathbb{Z}_n$, where $\tau = (\tau_0, \ldots, \tau_{q-1})$ such that $0 \leq \tau_j < 1$ for all j. Given $t = (t_0, \ldots, t_{q-1}) \in I^q$, let $\lfloor nt_0 \rfloor$ be the greatest integer less than or equal to nt_0 and define $\tau = (\tau_0, \ldots, \tau_{q-1})$ by $\tau_0 = nt_0 - \lfloor nt_0 \rfloor$ and $\tau_j = t_j$ otherwise. Let k be the element of $\mathbb{Z}_n$ that is congruent $\mod n$ to $z_0(A)\lfloor nt_0 \rfloor$. Now $P_q D_n t = P_q(\tau)$ and

$$p(\tfrac{1}{n} A_j D_n t) = p(\tfrac{1}{n} A_j \tau + \tfrac{1}{n} a_{j0} \lfloor nt_0 \rfloor).$$

If $\sigma_j = 0$ then $\frac{1}{n} a_{j0} \lfloor nt_0 \rfloor \in \mathbb{Z}$ by Theorem 3.1 so

$$p(\tfrac{1}{n} A_j D_n t) = p(\tfrac{1}{n} A_j \tau) = p(\tfrac{1}{n}(A_j \tau + k\sigma_j)).$$

If $\sigma_j = 1$ then, since $a_{j0} \equiv \zeta_0(A) \mod n$,

$$\begin{aligned} p(\tfrac{1}{n} A_j D_n t) &= p(\tfrac{1}{n} A_j \tau + \tfrac{1}{n} a_{j0} \lfloor nt_0 \rfloor) \\ &= p(\tfrac{1}{n} A_j \tau + \tfrac{1}{n} \zeta_0(A) \lfloor nt_0 \rfloor) \\ &= p(\tfrac{1}{n} A_j \tau + \tfrac{k}{n} \sigma_j). \end{aligned}$$

Therefore $P_{2q} g_{n,A,\sigma}(t) \in \Gamma(\Phi_{n,A,\sigma})$ and we have proved that P_{2q} maps $G_{n,A,\sigma}$ to $\Gamma(\Phi_{n,A,\sigma})$. To complete the proof of the claim, given

$$x = (P_q(\tau), P_q(\tfrac{1}{n}(A\tau + k\sigma)) \in \Gamma(\Phi_{n,A,\sigma})$$

we will define $t \in I^q$ such that $P_{2q} g_{n,A,\sigma}(t) = x$. Since $\zeta_0(A)$ and n are relatively prime, there exist integers α and β such that $\alpha\zeta_0(A) + \beta n = 1$. Given x above, let $t = (t_0, \ldots, t_{q-1}) \in I^q$ be defined by setting

$$t_0 = \frac{\tau_0 + [\alpha k]}{n},$$

where $[\alpha k] \in \mathbb{Z}_n$ such that $[\alpha k] \equiv \alpha k \bmod n$, and $t_i = \tau_i$ for all $i \geq 1$, so $P_q(D_n t) = P_q(\tau)$. Now

$$p(\tfrac{1}{n} A_j D_n t) = p(\tfrac{1}{n}(A_j \tau + a_{0j}[\alpha k])).$$

If $\sigma_j = 0$, then a_{0j} is divisible by n so

$$p(\tfrac{1}{n} A_j D_n t) = p(\tfrac{1}{n} A_j \tau) = p(\tfrac{1}{n}(A_j \tau + k\sigma_j)).$$

If $\sigma_j = 1$, then

$$p(\tfrac{a_{0j}[\alpha k]}{n}) = p(\tfrac{k}{n})$$

because $\zeta_0(A)\alpha k \equiv k \mod n$ and $a_{0j} \equiv \zeta_0(A) \mod n$ so $a_{0j}[\alpha k] \equiv k \mod n$ and again we have

$$p(\tfrac{1}{n}A_j D_n t) = p(\tfrac{1}{n}(A_j \tau + k\sigma_j)).$$

Thus we have shown that $P_{2q}g_{n,A,\sigma}(t) = x$ so $P_{2q}(G_{n,A,\sigma}) = \Gamma(\Phi_{n,A,\sigma})$ and therefore $G_{n,A,\sigma}$ is a lift of $\Gamma(\Phi_{n,A,\sigma})$ to the universal covering space of T^{2q}.

Consequently, $\Gamma(\Phi_{n,A,\sigma})$ is connected and, because it is an n-fold covering space of T^q, it is a compact connected q-dimensional abelian Lie group ([4], page 158) and therefore a q-torus ([7], Theorem 5.2(a), page 98). Define $\Pi_1, \Pi_2 \colon T^{2q} \to T^q$ to be the projections of the first q and last q factors, respectively, and define projections $\Pi'_1, \Pi'_2 \colon \mathbb{R}^{2q} \to \mathbb{R}^q$ in the same way, then $P_q\Pi'_1 = \Pi_1 P_{2q}$ and $P_q\Pi'_2 = \Pi_2 P_{2q}$. Now $\Pi'_1 g_{n,A,\sigma}(t) = D_n t$, that is, $\Pi'_1 g_{n,A,\sigma}$ is the restriction to $I^q \subseteq \mathbb{R}^q$ of the linear tranformation with matrix D_n. Thus the restriction $\Pi_1 \colon \Gamma(\Phi_{n,A,\sigma}) \to T^q$ is a map of q-tori that lifts to the linear transformation of the universal covering space with matrix D_n, so that matrix represents the fundamental group homomorphism induced by Π_1. Similarly, $\Pi'_2 g_{n,A,\sigma} \colon I^q \to \mathbb{R}^q$ is defined by $\Pi'_2 g_{n,A,\sigma}(t) = \frac{1}{n}AD_n t$ so the fundamental group homomorphism induced by the restriction $\Pi_2 \colon \Gamma(\Phi_{n,A,\sigma}) \to T^q$ is respresented by the matrix $\frac{1}{n}AD_n$. By Theorems 2.2 and 4.1,

$$\begin{aligned} N(\Phi_{n,A,\sigma}) &= N(\Pi_1^{\Phi_{n,A,\sigma}}, \Pi_2^{\Phi_{n,A,\sigma}}) \\ &= |\det(D_n - \tfrac{1}{n}AD_n)| \\ &= |\det(E - \tfrac{1}{n}A)\det(D_n)| \\ &= n|\det(E - \tfrac{1}{n}A)|. \end{aligned}$$

□

Corollary 4.1. *If $\Phi_{n,A,\sigma} \colon T^q \multimap T^q$ is a linear n-valued map where n is a prime, then $N(\Phi_{n,A,\sigma}) = n|\det(E - \frac{1}{n}A)|$.*

Example 4.1. *Let $\Phi_{3,A,\sigma} \colon T^4 \multimap T^4$ be the linear 3-valued map of Example 3.1 then, by the corollary,*

$$N(\Phi_{3,A,\sigma}) = 3|\det(E - \tfrac{1}{3}A)| = 30$$

Example 4.2. *There are no restrictions on the possible values of Nielsen numbers of linear n-valued maps of tori. For instance, given* *an integer* $m \geq 0$, *let*

$$A = \begin{bmatrix} 2-m & 1 \\ 0 & 4 \end{bmatrix}$$

then, by Theorem 3.1, there is a linear 2*-valued map* $\Phi_{2,A,\sigma}\colon T^2 \multimap T^2$ *for* $\sigma = (1,0)$*. By Corollary 4.1,*

$$N(\Phi_{2,A,\sigma}) = 2|\det(E - \tfrac{1}{2}A)| = m$$

The following proposition uses Theorem 4.2 to enlarge the class of linear n-valued maps of tori whose Nielsen numbers can be calculated.

Proposition 4.1. *Let* $\Phi_{n,A,\sigma}\colon T^q \multimap T^q$ *be a linear* n*-valued map and let* w *be the greatest common divisor of the set of integers* $\{n, \zeta_0(A), \ldots, \zeta_{q-1}(A)\}$*. If there exists* $\zeta_j(A)$ *such that* n/w *and* $\zeta_j(A)/w$ *are relatively prime, then* $N(\Phi_{n,A,\sigma}) = n|\det(E - \frac{1}{n}A)|$*.*

Proof. By definition, $\Phi_{n,A,\sigma}$ is obtained from $\widetilde{\Phi}_{n,A,\sigma}\colon I^q \multimap T^q$ where

$$\widetilde{\Phi}_{n,A,\sigma} = \{\widetilde{\Phi}^{(0)}_{n,A,\sigma}(t), \ldots, \widetilde{\Phi}^{(n-1)}_{n,A,\sigma}(t)\}$$

such that, for $k \in \mathbb{Z}_n$,

$$\widetilde{\Phi}^{(k)}_{n,A,\sigma}(t) = P_q(\tfrac{1}{n}(At + k\sigma)).$$

For $v \in \mathbb{Z}_w$, define $\widetilde{\Phi}^{\{v\}}\colon I^q \multimap T^q$ by

$$\widetilde{\Phi}^{\{v\}}(t) = \{\widetilde{\Phi}^{(v)}_{n,A,\sigma}(t), \widetilde{\Phi}^{(w+v)}_{n,A,\sigma}(t), \ldots, \widetilde{\Phi}^{((n/w-1)w+v)}_{n,A,\sigma}(t)\}.$$

In particular,

$$\widetilde{\Phi}^{\{0\}}(t) = \{\widetilde{\Phi}^{(0)}_{n,A,\sigma}(t), \widetilde{\Phi}^{(w)}_{n,A,\sigma}(t), \ldots, \widetilde{\Phi}^{(n-w)}_{n,A,\sigma}(t)\}$$

where, for $\ell \in \mathbb{Z}_{n/w}$,

$$\begin{aligned} \widetilde{\Phi}^{(\ell w)}_{n,A,\sigma}(t) &= (p(\tfrac{1}{n}(A_0 t + \ell w \sigma_0)), \ldots, p(\tfrac{1}{n}(A_{q-1}t + \ell w \sigma_{q-1}))) \\ &= P_q(\tfrac{1}{n}(At + \ell w \sigma)). \end{aligned}$$

Noting that

$$\tfrac{1}{n}(A_j t + \ell w \sigma_j) = \tfrac{w}{n}(\tfrac{1}{w}A_j t + \ell \sigma_j),$$

we have

$$\widetilde{\Phi}^{(\ell w)}_{n,A,\sigma}(t) = \widetilde{\Phi}^{(\ell)}_{\frac{n}{w}, \frac{1}{w}A, \sigma}(t)$$

and thus $\widetilde{\Phi}^{\{0\}}$ induces $\Phi^{\{0\}} = \Phi_{\frac{n}{w}, \frac{1}{w}A, \sigma}\colon T^q \multimap T^q$. Since $\zeta_j(A)/w = \zeta_j(\frac{1}{w}A)$ and n/w are relatively prime, Theorem 4.2 implies that

$\Gamma(\Phi^{\{0\}}) = \Gamma(\Phi_{\frac{n}{w},\frac{1}{w}A,\sigma})$ is a connected subset of T^{2q}. Given $r \in \mathbb{R}$ and $\sigma \in (\mathbb{Z}_2)^q$, we use the group operation on T^q to define a homeomorphism $h_{r\sigma}\colon T^q \to T^q$ by

$$h_{r\sigma}(\tau) = h_{r\sigma}(\tau_0, \dots, \tau_{q-1}) = (p(r\sigma_0)\tau_0, \dots, p(r\sigma_{q-1})\tau_{q-1})).$$

Now

$$p(\tfrac{1}{n}(A_j t + (\ell w + v)\sigma_j)) = p(\tfrac{v}{n}\sigma_j)p(\tfrac{1}{n}(A_j t + \ell w\sigma_j))$$

so $\widetilde{\Phi}^{\{v\}}$ induces $\Phi^{\{v\}}\colon T^q \multimap T^q$ such that

$$\Phi^{\{v\}} = h_{\frac{v}{n}\sigma}\Phi^{\{0\}} = h_{\frac{v}{n}\sigma}\Phi_{\frac{n}{w},\frac{1}{w}A,\sigma}$$

where, if $\Phi^{\{0\}}(\tau) = \{y_0, \dots, y_{\frac{n}{w}-1}\}$, then

$$h_{\frac{v}{n}\sigma}\Phi^{\{0\}}(\tau) = \{h_{\frac{v}{n}\sigma}(y_0), \dots, h_{\frac{v}{n}\sigma}(y_{\frac{n}{w}-1})\}.$$

Therefore $\Gamma(\Phi_{n,A,\sigma})$ has w components: the $\Gamma(\Phi^{\{v\}})$. For any $v \in \mathbb{Z}_w$ and the restrictions $\Pi_1^{\{v\}}, \Pi_2^{\{v\}}\colon \Gamma(\Phi^{\{v\}}) \to T^q$ of the projections of the first q and last q factors of T^{2q}, we have $\Pi_1^{\{v\}} = \Pi_1^{\{0\}}$ and $\Pi_2^{\{v\}} = h_{\frac{v}{n}\sigma}\Pi_2^{\{0\}}$ so, since $h_{\frac{n}{v}\sigma}$ is a homeomorphism, $N(\Pi_1^{\{v\}}, \Pi_2^{\{v\}}) = N(\Pi_1^{\{0\}}, \Pi_2^{\{0\}})$. By Theorems 2.2 and 4.2,

$$N(\Pi_1^{\{0\}}, \Pi_2^{\{0\}}) = N(\Phi_{\frac{n}{w},,\frac{1}{w}A,\sigma}) = \tfrac{n}{w}|\det(E - \tfrac{w}{n}(\tfrac{1}{w}A))|.$$

Therefore,

$$N(\Phi_{n,A,\sigma}) = w(\tfrac{n}{w}|\det E - \tfrac{1}{n}A)|) = n|\det(E - \tfrac{1}{n}A)|.$$

□

Example 4.3. *There is a linear* 8*-valued map* $\Phi_{8,A,\sigma}\colon T^3 \multimap T^3$ *where*

$$A = \begin{bmatrix} 4 & -2 & -12 \\ 12 & 6 & 4 \\ -4 & -10 & 4 \end{bmatrix}$$

and $\sigma = (1,1,1)$ *with* $\zeta(A) = (4,6,4)$*. Thus the greatest common divisor of the set* $\{n, \zeta_0(A), \zeta_1(A), \zeta_2(A)\}$ *is* $w = 2$*. Since* $n/w = 4$ *and* $\zeta_1(A)/w = 3$ *are relatively prime, we have*

$$N(\Phi_{n,A,\sigma}) = 8|\det(E - \tfrac{1}{8}A)| = 20.$$

Corollary 4.2. *Let* $\Phi_{n,A,\sigma}\colon T^q \multimap T^q$ *be a linear* n*-valued map such that* $\zeta_0(A) = \zeta_1(A) = \cdots = \zeta_{q-1}(A)$*, then* $N(\Phi_{n,A,\sigma}) = n|\det(E - \frac{1}{n}A)|$.

Proof. Let w be the greatest common divisor of the set of integers $\{n, \zeta_0(A), \dots, \zeta_{q-1}(A)\}$, then w is the greatest common divisor of n and $\zeta_0(A)$ so n/w and $\zeta_0(A)/w$ are relatively prime and the result follows by Proposition 4.1. $\square$

Remark 1. *We note that if a matrix A satisfies the conditions of Theorem 3.1, then $n|\det(E - \frac{1}{n}A)|$ is an integer. Moreover, if $\Phi_{n,A,\sigma}\colon T^2 \multimap T^2$ is a linear n-valued map where*

$$A = \begin{bmatrix} a_{00} & 0 \\ a_{10} & a_{11} \end{bmatrix}$$

and $\sigma_0 = 1$, then $\Phi_{n,A,\sigma}$ is an n-valued fiber map with respect to the product fibration $\Pi_1\colon T^2 \to S^1$. The induced n-valued map of the base is of degree a_{00} and $\Phi_{n,A,\sigma}$ restricts on each fixed fiber to a single-valued map of degree a_{11}/n. Therefore, by Theorem 5.1 of [1] and Theorem 4.1 of [2]

$$N(\Phi_{n,A,\sigma}) = |n - a_{00}||1 - \tfrac{a_{11}}{n}| = n|\det(E - \tfrac{1}{n}A)|$$

for any *value of $\zeta_0(A)$. These observations together with the results of this section lead us to conjecture that*

$$N(\Phi_{n,A,\sigma}) = n|\det(E - \tfrac{1}{n}A)|$$

holds for all linear n-valued maps $\Phi_{n,A,\sigma}\colon T^q \multimap T^q$.

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