

THE LEFSCHETZ NUMBER OF AN n -VALUED MULTIMAP

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Abstract

An n -valued multimap is a continuous multivalued function $\phi: X \multimap Y$ such that $\phi(x)$ is an unordered subset of n points of Y for each $x \in X$. If X and Y are finite polyhedra, then ϕ induces a graded homomorphism of homology with rational coefficients. For $\phi: X \multimap X$ the Lefschetz number $L(\phi)$ of ϕ is defined to be the Lefschetz number of the induced homomorphism. If $L(\phi) \neq 0$, then every n -valued multimap homotopic to ϕ has a fixed point. If X is the circle, then the Lefschetz number of ϕ is related to the Nielsen number $N(\phi)$ of Schirmer as in the single-valued case, that is, $N(\phi) = |L(\phi)|$.

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1 Introduction

In 1957, O'Neill [4] introduced a very general concept of induced homology homomorphism for multivalued functions. If X and Y are compact metric spaces then, for any multivalued function $\phi: X \multimap Y$, there is a vector space of graded homomorphisms

$$h = \{h_k: H_k(X) \rightarrow H_k(X)\}$$

of homology groups with coefficients in a field, all of which he considered induced homomorphisms of ϕ . When ϕ is a single-valued continuous function, this vector space consists of the scalar multiples of the usual induced homology homomorphism. If X is a finite polyhedron, so the homology with rational coefficients $H_*(X)$ is finite-dimensional, and $h = \{h_k: H_k(X) \rightarrow H_k(X)\}$ is a graded homomorphism, then its *Lefschetz number* $\Lambda(h)$ is defined by

$$\Lambda(h) = \sum_k (-1)^k \text{trace}(h_k).$$

Note that if r is a scalar, then the Lefschetz number of the graded homomorphism rh has the property $\Lambda(rh) = r\Lambda(h)$. Thus, in O’Neill’s theory, no specific value called a Lefschetz number can be assigned to a function.

A *multimap* $\phi: X \multimap Y$ is a multivalued function that is continuous, that is, both upper and lower semi-continuous [1]. O’Neill’s version of the Lefschetz fixed point theorem ([4], Theorem 9) states that if $\phi: X \multimap X$ is a multimap of a finite polyhedron such that, for each $x \in X$, $\phi(x)$ is either homologically trivial or consists of exactly n homologically trivial components and $\Lambda(h) \neq 0$ for some induced homology homomorphism h , then ϕ has a fixed point, that is, $x \in \phi(x)$ for some $x \in X$.

In 1984, Schirmer [5] initiated a study of multimaps $\phi: X \multimap Y$ of finite polyhedra that are *n-valued*, that is, $\phi(x)$ is an unordered set of exactly n points of Y , for each $x \in X$. For this special case of the class of multivalued functions considered by O’Neill, she proved a simplicial approximation theorem ([5], Theorem 4) and remarked that it “could, in fact, be used to define for an n -valued continuous multifunction ϕ an induced homomorphism $\phi_*: H_*(X) \rightarrow H_*(Y)$ in a way different from the one given by O’Neill, but we do not pursue (sic) this topic”.

We will show how Schirmer’s simplicial approximation theorem for an n -valued multimap $\phi: X \multimap Y$ leads to a well-defined homomorphism of rational homology $\phi_*: H_*(X) \rightarrow H_*(Y)$. There is therefore associated to each n -valued multimap $\phi: X \multimap X$ on a finite polyhedron a *unique* Lefschetz number defined by $L(\phi) = \Lambda(\phi_*)$. We will prove that the induced homomorphism, and therefore the Lefschetz number $L(\phi)$, is an invariant of the n -valued homotopy class of ϕ and that $L(\phi) \neq 0$ implies that ϕ has a fixed point. Moreover, we will demonstrate that, for n -valued multimaps of the circle, this Lefschetz number is related to the Nielsen number for n -valued multimaps developed by Schirmer in [6] and [7] in the same way that, for single-valued maps, the Lefschetz number is related to the classical Nielsen number. That is, letting $\phi: S^1 \multimap S^1$ be an n -valued multimap of the circle, then the Nielsen number $N(\phi)$ of Schirmer is equal to the absolute value of $L(\phi)$.

2 The induced homomorphism and the Lefschetz number

Given an n -valued multimap $\phi: X \multimap Y$ of finite polyhedra. For each $x \in X$, define $\gamma(x; \phi)$ to be the minimum distance $d(y_i, y_j)$

among all pairs of points $y_i, y_j \in \phi(x)$. Let $\gamma(\phi)$ denote the infimum of $\gamma(x; \phi)$ for $x \in X$. It is noted in [5] that the continuity of ϕ , and hence of $\gamma(\cdot; \phi)$, together with the compactness of X implies that $\gamma(\phi) > 0$.

The Splitting Lemma ([5], Lemma 1) implies that if S is a closed, simply-connected subset of S of X , then the restriction of ϕ to S splits, that is, there are single-valued maps $f_1, \dots, f_n: S \rightarrow Y$ such that $\phi(x) = \{f_1(x), \dots, f_n(x)\}$ for all $x \in S$. Then, fixing simplicial structures $X = |K|$ and $Y = |L|$, an n -valued multimap $\phi': X \multimap Y$ is said to be *simplicial* if, for each simplex σ of K , the restriction of ϕ' to the polyhedron whose geometric realization $\bar{\sigma}$ is the closure of $\sigma \in K$, splits into n maps $f_j: |\bar{\sigma}| \rightarrow |L|$ each of which is simplicial. Suppose $\epsilon > 0$ is given. The Simplicial Approximation Theorem of [5] (Theorem 4), states that if the mesh of a subdivision $|L'|$ of $|L|$ is less than both $\frac{1}{4}\gamma(\phi)$ and ϵ , then there is a subdivision K' of K and a simplicial n -valued multimap $\phi': |K'| \multimap |L'|$ such that the Hausdorff distance $d(\phi(x), \phi'(x)) < \epsilon$ for all $x \in X$. Moreover, as noted on page 79 of [5], for each $\sigma \in K'$, the image

$$\phi'(\bar{\sigma}) = \bar{\tau}_1 \cup \dots \cup \bar{\tau}_n$$

where the $\bar{\tau}_j$ are n disjoint closed simplices of $|L'|$.

Now let $\{C_k(K')\}$ and $\{C_k(L')\}$ denote the rational simplicial chain complexes and define $\phi_k: C_k(K') \rightarrow C_k(L')$ by setting

$$\phi_k(\sigma) = \tau_1 + \dots + \tau_n.$$

Since each $\bar{\tau}_j = f_j(\bar{\sigma})$ and the f_j induce chain maps, then $\{\phi_k\}$ is a chain map of these chain complexes. Moreover, since an n -valued homotopy of a split n -valued map is also split ([3], Theorem 2.1), an n -valued homotopy of ϕ to $\psi: X \multimap Y$, when restricted to $\bar{\sigma}$, splits into n homotopies of the f_j which determines a splitting g_1, \dots, g_n of the restriction of ψ to $\bar{\sigma}$. Those homotopies define chain homotopies D_1, \dots, D_n and thus, defining

$$D(\sigma) = D_1(\sigma) + \dots + D_n(\sigma),$$

we obtain a chain homotopy between $\{\phi_k\}$ and $\{\psi_k\}$. The chain map $\{\phi_k: C_k(K') \rightarrow C_k(L')\}$ thus induces a homomorphism of rational homology

$$\phi_* = \{\phi_{*k}: H_k(X) \rightarrow H_k(Y)\}$$

such that if $\phi, \psi: X \multimap Y$ are homotopic n -valued multimaps, then $\phi_* = \psi_*$.

Now suppose $\phi: X \rightarrow X$ is an n -valued multimap of a finite polyhedron. The definition identified a specific homomorphism of

rational homology ϕ_* as a finite set of nontrivial linear endomorphisms, so the definition

$$L(\phi) = \Lambda(\phi_*) = \sum_k (-1)^k \text{trace}(\phi_{*k}).$$

determines a well-defined *Lefschetz number* for ϕ .

Theorem 1. (*Lefschetz fixed point theorem for n -valued multimaps*)
If $\phi: X \rightarrow X$ is an n -valued multimap of a finite polyhedron such that $L(\phi) \neq 0$, then $x \in \phi(x)$ for some $x \in X$.

Proof. We will prove that if ϕ has no fixed points, then $L(\phi) = 0$. For each $x \in X$, let $\epsilon(x; \phi)$ be the minimum of the distance from x to each point $y_j \in \phi(x)$. Let $\epsilon(\phi)$ be the infimum of $\epsilon(x; \phi)$ for all $x \in X$. If ϕ has no fixed points, then the continuity of ϕ , and thus of $\epsilon(\cdot; \phi)$ together with the compactness of X imply that $\epsilon(\phi) > 0$. Let $X = |K|$ be a triangulation of X of mesh smaller than both $\frac{1}{4}\epsilon(\phi)$ and $\frac{1}{2}\epsilon(\phi)$ and let $\phi': |K'| \rightarrow |K|$ be a simplicial approximation to ϕ . Let $\sigma \in K'$ be a k -simplex then, as above, we have the chain map $\phi_k(\sigma) = \tau_1 + \cdots + \tau_n$ where $\bar{\tau}_j = f_j(\bar{\sigma})$ for a splitting f_1, \dots, f_n of the restriction of ϕ' to $\bar{\sigma}$. As in the proof of Theorem II.A.1 on page 26 of [2], it must be that $\bar{\sigma} \cap \bar{\tau}_j = \emptyset$ for all $j = 1, \dots, n$. Moreover, there is a chain map $h = \{h_k\}$ obtained from barycentric subdivision such that the diagonal entries of the matrix of $\phi_k h_k: C_k(K) \rightarrow C_k(K)$ are all zero so $\text{trace}(\phi_k h_k) = 0$. The Hopf Trace Theorem ([2], Theorem I.D.3) then completes the proof that $L(\phi) = 0$. \square

3 The circle

In the case $n = 1$, the definition of the Lefschetz number and the proof of the Lefschetz theorem reduce to the classical ones. However, there might be ways to extend Lefschetz theory to n -valued multimaps with $n > 1$ other than that described in the previous section. As evidence that the generalization we have presented is the appropriate one, we will demonstrate that a key relationship between the classical Lefschetz and Nielsen fixed point theories holds also for this Lefschetz number and the Nielsen number for n -valued multimaps that was developed by Schirmer in ([6], [7]).

Theorem 2. *Let $\phi: S^1 \rightarrow S^1$ be an n -valued multimap of the circle with Nielsen number $N(\phi)$ and Lefschetz number $L(\phi)$, then $N(\phi) = |L(\phi)|$.*

Proof. Let $p: \mathbb{R} \rightarrow S^1$ be the map defined by $p(t) = e^{i2\pi t}$. For an integer d , define an n -valued multimap $\phi_{n,d}: S^1 \multimap S^1$ by

$$\phi_{n,d}(p(t)) = \left\{ p\left(\frac{d}{n}t\right), p\left(\frac{d}{n}t + \frac{1}{n}\right), \dots, p\left(\frac{d}{n}t + \frac{n-1}{n}\right) \right\}.$$

By Theorem 3.1 of [3], the map ϕ is homotopic to $\phi_{n,d}$ for some d . Since the Lefschetz number is homotopy invariant and, by Theorem 6.5 of [6], the Nielsen number is also homotopy invariant, it is sufficient to prove that $N(\phi_{n,d}) = |L(\phi_{n,d})|$ for all positive integers n and integers d .

For $r \geq 2$, let S_r^1 denote the triangulation of S^1 with vertex set $\{v_j^r = p(\frac{j}{r})\}$ for $j = 0, 1, \dots, r-1, r$. Note that $v_r^r = v_0^r$. The group $C_1(S_r^1)$ of simplicial 1-chains with rational coefficients is generated by the 1-simplices $\{[v_j^r, v_{j+1}^r]\}$. The homology class \bar{z} of the cycle

$$z_r = [v_0^r, v_1^r] + [v_1^r, v_2^r] + \dots + [v_{r-1}^r, v_r^r]$$

generates $H_1(S^1)$. The multimap $\phi_{n,d}: S_{n|d}^1 \multimap S_{n^2}^1$ is simplicial because, for $0 \leq j \leq n|d| - 1$, we have

$$\phi_{n,d}(v_j^{n|d|}) = \{v_{\sigma(j)}^{n^2}, v_{\sigma(j)+n}^{n^2}, v_{\sigma(j)+2n}^{n^2}, \dots, v_{\sigma(j)+(n-1)n}^{n^2}\}$$

where $\sigma(j) = j$ if $d \geq 0$ and $\sigma(j) = -j$ if $d < 0$. The sets $\phi_{n,d}(v_j^{n|d|})$ and $\phi_{n,d}(v_{j'}^{n|d|})$ are identical if and only if j and j' are congruent mod n . Otherwise they are disjoint. The set of vertices of $S_{n|d|}^1$ is thus partitioned, in terms of the images under $\phi_{n,d}$, into n subsets, each consisting of $|d|$ vertices.

If $d = 0$, then $\phi_{n,d}$ is a constant multimap so it induces the trivial homomorphism $\phi_{n,d*}: H_1(S^1) \rightarrow H_1(S^1)$. Represent the conjugacy class mod n of an integer j by $\langle j \rangle$ where $0 \leq \langle j \rangle \leq n-1$. If $d > 0$, then the chain map $C_1(\phi_{n,d}): C_1(S_{nd}^1) \rightarrow C_1(S_{n^2}^1)$ is given by

$$C_1(\phi_{n,d})[v_j^{nd}, v_{j+1}^{nd}] = \sum_{k=0}^{n-1} [v_{\langle j \rangle + kn}^{n^2}, v_{\langle j \rangle + kn + 1}^{n^2}]$$

and therefore

$$C_1(\phi_{n,d})(z_{nd}) = \sum_{j=0}^{nd-1} \sum_{k=0}^{n-1} [v_{\langle j \rangle + kn}^{n^2}, v_{\langle j \rangle + kn + 1}^{n^2}] = dz_{n^2}$$

which implies that $\phi_{n,d*}(\bar{z}) = d\bar{z}$.

If $d < 0$, then

$$\begin{aligned} C_1(\phi_{n,d})[v_j^{n|d|}, v_{j+1}^{n|d|}] &= \sum_{k=0}^{n-1} [v_{\langle -j \rangle + kn}^{n^2}, v_{\langle -j \rangle + kn-1}^{n^2}] \\ &= \sum_{k=0}^{n-1} -[v_{\langle -j \rangle + kn-1}^{n^2}, v_{\langle -j \rangle + kn}^{n^2}]. \end{aligned}$$

Each generator of $C_1(S_n^1)$ appears in the chain $C_1(\phi_{n,d})(z_{n|d|})$ exactly $|d|$ times, so

$$C_1(\phi_{n,d})(z_{n|d|}) = -|d|z_{n^2} = dz_{n^2}$$

which implies that $\phi_{n,d*}(\bar{z}) = d\bar{z}$ in this case also.

Letting $1 \in H_0(S^1)$ be a generator, it is clear from the definition that $\phi_*(1) = n1$ for any n -valued multimap $\phi: S^1 \multimap S^1$. Thus $L(\phi_{n,d}) = n-d$ for all positive integers n and integers d . By Theorem 5.1 of [3], $N(\phi_{n,d}) = |n-d|$ so $N(\phi_{n,d}) = |L(\phi_{n,d})|$ and the proof is complete. \square

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