1. Determine whether the series is convergent or divergent.

$$(a)\quad \sum_{n=3}^{\infty} \frac{\sqrt{n}}{n^2 - 2n} \qquad\qquad (b)\quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/3}}$$

[10 points] (a)

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2-2n}}{\frac{\sqrt{n}}{n^2}} = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{n^2-2n}\,\frac{n^2}{\cancel{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2-2n}\,\frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{1}{1-2/n} = 1 \text{ and}$$

$\sum_{n=3}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=3}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, so converges by the Limit Comparison Test

[10 points] (b)

$$\int \frac{dx}{x(\ln x)^{1/3}} = \int u^{-1/3}\,du = \frac{3}{2}u^{2/3} + C$$

$$u = \ln x \quad du = \frac{dx}{x} \qquad = \frac{3}{2}(\ln x)^{2/3} + C$$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^{1/3}} = \lim_{R\to\infty} \frac{3}{2}(\ln x)^{2/3}\Big|_2^R$$

$$= \lim_{R\to\infty} \frac{3}{2}(\ln R)^{2/3} - \frac{3}{2}(\ln 2)^{2/3} = \infty$$

so diverges by the Integral Test

2. Find the interval on which

$$f(x) = \left(\frac{x}{3}\right)^{1/x} \qquad x > 0$$

is increasing.

$$y = \left(\frac{x}{3}\right)^{1/x} = \left(\frac{x}{3}\right)^{x^{-1}}$$

$$\ln y = x^{-1} \ln\left(\frac{x}{3}\right)$$

$$\frac{y'}{y} = -x^{-2} \ln\left(\frac{x}{3}\right) + x^{-1} \frac{1/3}{x/3}$$

$$= -x^{-2} \ln\left(\frac{x}{3}\right) + x^{-2}$$

$$= x^{-2}\left(1 - \ln\left(\frac{x}{3}\right)\right)$$

$$y' = x^{-2}\left(1 - \ln\left(\frac{x}{3}\right)\right)\left(\frac{x}{3}\right)^{1/x} > 0$$

if $1 - \ln\left(\frac{x}{3}\right) > 0$ so $\ln\left(\frac{x}{3}\right) < 1$

$\frac{x}{3} < e$ and $x < 3e$ so the

interval is $(0, 3e)$

3. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n(x-1)^n}{\sqrt{2n+1}}$$

$$\lim_{n\to\infty} \frac{2^{n+1}}{\sqrt{2(n+1)+1}} \frac{\sqrt{2n+1}}{2^n} = \lim_{n\to\infty} 2\sqrt{\frac{2n+1}{2n+3}} = 2$$

$2|x-1| < 1 \quad |x-1| < \frac{1}{2}$ so $\frac{1}{2} < x < \frac{3}{2}$

$x = \frac{1}{2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}-1\right)^n}{\sqrt{2n+1}} = \sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{\sqrt{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$

converges by Leibniz Test

$x = \frac{3}{2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{\sqrt{2n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$ diverges by the

Comparison test since $\frac{1}{\sqrt{2n+1}} > \frac{1}{n}$

interval of convergence $\frac{1}{2} \le x < \frac{3}{2}$

4. Evaluate the integral

$$\int_1^\infty \frac{\ln(x^4)}{x^3}\,dx$$

$$\int \frac{\ln(x^4)}{x^3}\,dx = -\frac{1}{2}x^{-2}\ln(x^4) + 2\int x^{-3}\,dx$$

$u = \ln(x^4)$ $\qquad dv = x^{-3}\,dx$

$du = \frac{4x^3}{x^4}\,dx = 4x^{-1}\,dx$ $\qquad v = -\frac{1}{2}x^{-2}$

$$= -\frac{1}{2}x^{-2}\ln(x^4) + 2\left(-\frac{1}{2}x^{-2}\right)$$

$$\int_1^\infty \frac{\ln(x^4)}{x^3}\,dx = \lim_{R\to\infty} -\frac{\ln(x^4)}{2x^2} - \frac{1}{x^2}\Big|_1^R$$

$$= \lim_{R\to\infty} -\frac{\ln(R^4)}{2R^2} - \frac{1}{R^2} - (0-1) = 1$$

because

$$\lim_{R\to\infty} \frac{\ln(R^4)}{2R^2} \overset{H}{=} \lim_{R\to\infty} \frac{\frac{4R^3}{R^4}}{4R}$$

$$= \lim_{R\to\infty} \frac{4}{R^2} = 0$$

5. Find the sum of the series

$$(a)\quad \sum_{n=2}^{\infty} e^{5-3n} \qquad\qquad (b)\quad \sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$$

(Hint for part (b): Write the partial sum S_3.)

[10 points] (a) $\displaystyle\sum_{n=2}^{\infty} e^{5-3n} = \sum_{n=2}^{\infty} \frac{e^5}{e^{3n}}$

$$= \sum_{n=2}^{\infty} e^5\left(\frac{1}{e^3}\right)^n = \frac{e^5\left(\frac{1}{e^3}\right)^2}{1-\frac{1}{e^3}} = \frac{\frac{1}{e}}{1-\frac{1}{e^3}} = \frac{e^2}{e^3-1}$$

[10 points] (b) $S_3 = \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{9}\right) = \frac{1}{3}-\frac{1}{9}$

$$S_N = \frac{1}{3} - \frac{1}{2N+3}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right) = \lim_{N\to\infty} S_N = \frac{1}{3}$$

6. (a) Write the Taylor polynomial $T_3(x)$ for $f(x) = (x-1)^{-2}$ at $x = 2$. (b) Find the Taylor series for $f(x) = (x-1)^{-2}$ at $x = 2$. (c) Find the radius of convergence of the series of part (b).

[7 points] (a) $f'(x) = -2(x-1)^{-3}$ $f''(x) = (-2)(-3)(x-1)^{-4}$

$f'''(x) = (-2)(-3)(-4)(x-1)^{-5}$ $f(2) = 1$ $f'(2) = -2$

$f''(2) = 6$ $f'''(2) = -24$

$T_3(x) = 1 - 2(x-2) + \frac{6}{2!}(x-2)^2 - \frac{24}{3!}(x-2)^3$

[8 points] (b) $f^{(n)}(x) = (-1)^n (n+1)!\,(x-1)^{-(n+2)}$

$f^{(n)}(2) = (-1)^n (n+1)!$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} (x-2)^n = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-2)^n$$

[5 points] (c) $\lim_{n\to\infty} \frac{n+2}{n+1} = 1$ so $R = 1$

7. Evaluate the integrals

$$(a) \quad \int \frac{x+1}{x^3+x}\,dx \qquad\qquad (b) \quad \int \frac{dx}{\sqrt{2-x^2}}$$

[13 points] (a) $\dfrac{x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$= \dfrac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$

$A+B=0 \quad C=1 \quad A=1 \quad B=-1$

$\int \dfrac{x+1}{x^3+x}\,dx = \int \dfrac{1}{x} + \dfrac{-x+1}{x^2+1}\,dx$

$= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx + \int \frac{dx}{x^2+1} = \ln|x| - \frac{1}{2}\int \frac{du}{u} + \tan^{-1}x + C$

$u = x^2+1 \quad du = 2x\,dx$

$x\,dx = \frac{1}{2}\,du$

$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$

[7 points] (b) $\int \dfrac{dx}{\sqrt{2-x^2}} = \int \dfrac{dx}{\sqrt{2}\sqrt{1-\frac{x^2}{2}}}$

$= \frac{1}{\sqrt{2}}\int \dfrac{dx}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} = \cancel{\frac{1}{\sqrt{2}}}\int \dfrac{\cancel{\sqrt{2}}\,du}{\sqrt{1-u^2}} = \sin^{-1}u + C$

$u = \frac{x}{\sqrt{2}} \quad du = \frac{1}{\sqrt{2}}\,dx$

$= \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

8. Determine whether the series converges absolutely, converges conditionally or diverges.

$$(a) \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{\log_2 n} \qquad\qquad (b) \quad \sum_{n=1}^{\infty} \left(\frac{n}{1-2n}\right)^n$$

[10 points] (a) $\dfrac{1}{\log_2 n} = \dfrac{\ln 2}{\ln n} > \dfrac{1}{n}$

so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log_2 n}$ diverges but $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\log_2 n}$ converges by Leibniz' Test so conditionally convergent

[10 points] (b) $\left| \left(\dfrac{n}{1-2n}\right)^n \right| = \left(\dfrac{n}{2n-1}\right)^n$

$\displaystyle\lim_{n\to\infty} \left(\left(\frac{n}{2n-1}\right)^n \right)^{1/n} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} < 1$

so converges absolutely by the Root Test

9. Use the error bound for the Midpoint Rule to determine how large N must be to approximate

$$\int_0^1 e^{x^3}\,dx$$

by M_N to within 10^{-5}. (Leave your answer in the form $N =$ an expression that could be computed with a calculator.)

$f(x) = e^{x^3}$ $f'(x) = e^{x^3}(3x^2)$

$f''(x) = e^{x^3}(3x^2)^2 + e^{x^3}(6x) > 0$

and increasing since $f'''(x) > 0$

so $K = f''(1) = e(3)^2 + e(6) = 15e$

$$\frac{15e(1-0)^3}{24N^2} < 10^{-5}$$

$$\frac{15e}{24} < \frac{N^2}{10^5} \qquad N^2 > \frac{15e\,10^5}{24}$$

$$N > \left(\frac{15e\,10^5}{24}\right)^{1/2}$$

10. Determine all the integers (that is, whole numbers) k for which the series

$$\sum_{n=1}^{\infty} \frac{(n!)^k}{(2n)!}$$

converges, showing your work.

$$\lim_{n\to\infty} \frac{((n+1)!)^k}{(2(n+1))!} \frac{(2n)!}{(n!)^k}$$

$$= \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} \left(\frac{(n+1)!}{n!}\right)^k$$

$$= \lim_{n\to\infty} \frac{(n+1)^k}{(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^k}{4n^2+6n+2} = \begin{cases} 1/4 & \text{if } k=2 \\ 0 & \text{if } k<2 \\ \infty & \text{if } k>2 \end{cases}$$

So converges if $k \le 2$