1. Differentiate

$$(a)\ f(x) = \cos^{-1}\left(\frac{x}{2^x+1}\right) \qquad (b)\ f(x) = (\log_3 x)^{\sin x}$$

(a) [10 points]

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{x}{2^x+1}\right)^2}} \frac{(2^x+1) - x((\ln 2)2^x)}{(2^x+1)^2}$$

(b) [10 points]

$$\ln(f(x)) = \sin x \ln(\log_3 x)$$

$$\frac{f'(x)}{f(x)} = \cos x \ln(\log_3 x) + \sin x \left(\frac{\frac{1}{\ln 3\, x}}{\log_3 x}\right)$$

$$f'(x) = \left(\cos x \ln(\log_3 x) + \sin x \left(\frac{\frac{1}{\ln 3\, x}}{\log_3 x}\right)\right)(\log_3 x)^{\sin x}$$

2. Write the volume of the solid of revolution obtained by rotating the region of the first quadrant bounded by the y-axis, the curve $y = \sinh x$ and the curve

$$y = 1 - \frac{e^{-x}}{2}$$

about the y-axis as a definite integral. Do NOT attempt to evaluate the integral.

$$1 - \frac{e^{-x}}{2} = \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$2 - e^{-x} = e^{x} - e^{-x}$$

$$e^{x} = 2 \quad \text{so} \quad x = \ln 2$$

$$\text{Volume} = \int_0^{\ln 2} 2\pi x \left(\left(1 - \frac{e^{-x}}{2}\right) - \sinh x \right) dx$$

3. Calculate the limits

$$(a)\ \lim_{x\to\infty} x^{\ln 2/(1+\ln x)} \qquad\qquad (b)\ \lim_{x\to 0}\left(\frac{e^x}{e^x-1}-\frac{1}{x}\right)$$

(a) [10 points]

$$y = x^{\ln 2/(1+\ln x)} \qquad \ln y = \left(\frac{\ln 2}{1+\ln x}\right)\ln x$$

$$\lim_{x\to\infty}\frac{(\ln 2)\ln x}{1+\ln x} \underset{H}{=} \lim_{x\to\infty}\frac{(\ln 2)\cancel{x^{-1}}}{\cancel{x^{-1}}} = \ln 2$$

$$\lim_{x\to\infty} x^{\ln 2/(1+\ln x)} = e^{\ln 2} = 2$$

(b) [10 points]

$$\lim_{x\to 0}\left(\frac{e^x}{e^x-1}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{xe^x-(e^x-1)}{(e^x-1)x}$$

$$\underset{H}{=} \lim_{x\to 0}\frac{e^x+xe^x-e^x}{e^x x+(e^x-1)} = \lim_{x\to 0}\frac{xe^x}{xe^x+e^x-1}$$

$$\underset{H}{=} \lim_{x\to 0}\frac{e^x+xe^x}{e^x+xe^x+e^x} = \frac{1}{2}$$

4. Find the absolute maximum and the absolute minimum of the function

$$f(x) = \frac{\ln x}{x^2}$$

on the interval $[\frac{1}{e}, e]$.

$$f'(x) = \frac{x^{-1}x^2 - (\ln x)(2x)}{x^4}$$

$$= \frac{x - 2x\ln x}{x^4} = \frac{x(1-2\ln x)}{x^4}$$

$$= \frac{1-2\ln x}{x^3} = 0$$

$1-2\ln x = 0 \quad \ln x = \frac{1}{2} \quad x = e^{1/2}$

$$f(e^{1/2}) = \frac{\ln(e^{1/2})}{(e^{1/2})^2} = \frac{1/2}{e} = \frac{1}{2e} \quad \text{max}$$

$$f\left(\frac{1}{e}\right) = f(e^{-1}) = \frac{\ln(e^{-1})}{(e^{-1})^2} = \frac{-1}{\frac{1}{e^2}} = -e^2 \quad \text{min}$$

$$f(e) = \frac{\ln(e)}{e^2} = \frac{1}{e^2}$$

5. Calculate the indefinite integrals

$$(a)\ \int \tan^{-1} x\, dx \qquad\qquad (b)\ \int x \log_3\left(\frac{x}{x^2+2}\right) dx$$

(a) [7 points]

$u = \tan^{-1}x \quad du = \frac{1}{x^2+1}dx \quad dv = dx \quad v = x$

$\int \tan^{-1}x\, dx = x\tan^{-1}x - \int \frac{x}{x^2+1}dx \qquad w = x^2+1$

$dw = 2x\,dx$

$x\,dx = \tfrac{1}{2}dw$

$= x\tan^{-1}x - \int \frac{1}{w}\frac{1}{2}dw = x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + C$

(b) [13 points]

$\int x\log_3\left(\frac{x}{x^2+2}\right)dx = \frac{1}{\ln 3}\left(\int x\ln x\,dx - \int x\ln(x^2+2)dx\right)$

$u = \ln x \quad du = x^{-1}dx \qquad w = x^2+2 \quad dw = 2x\,dx$

$dv = x\,dx \quad v = \frac{1}{2}x^2 \qquad x\,dx = \frac{1}{2}dw$

$= \frac{1}{\ln 3}\left(\frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx - \int \ln w\,\frac{1}{2}dw\right)$

$= \frac{1}{\ln 3}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 - \frac{1}{2}(w\ln w - w)\right) + C$

$= \frac{1}{\ln 3}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 - \frac{1}{2}((x^2+2)\ln(x^2+2) - (x^2+2)\right) + C$