1. Calculate the arc length of the curve $y = 2(x+1)^{3/2}$ over the interval $[0, 1]$.

$y' = 3(x+1)^{1/2}$ $\quad (y')^2 = 9(x+1)$

arc length $= \int_0^1 \sqrt{9x+10}\, dx$

$u = 9x+10 \quad du = 9\,dx \quad dx = \frac{1}{9} du$

$= \int_{10}^{19} u^{1/2} \frac{1}{9} du = \frac{1}{9} \frac{2}{3} u^{3/2} \Big|_{10}^{19}$

$= \frac{2}{27}\left(19^{3/2} - 10^{3/2}\right)$

2. (a) Evaluate

[10 points]

$$\int \frac{6\,dx}{(x+1)(x+3)}$$

(b) Using part (a), evaluate

[10 points]

$$\int_0^\infty \frac{6\,dx}{(x+1)(x+3)}$$

(a) $\frac{6}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3} = \frac{A(x+3)+B(x+1)}{(x+1)(x+3)}$

$6 = (A+B)x + (3A+B) \quad A+B=0 \quad B=-A$

$3A - A = 6 \quad A = 3, \; B = -3$

$$\int \frac{6\,dx}{(x+1)(x+3)} = \int \frac{3}{x+1} - \frac{3}{x+3}\,dx = 3\ln|x+1| - 3\ln|x+3| + C$$

(b) $$\int_0^\infty \frac{6\,dx}{(x+1)(x+3)} = \lim_{R\to\infty} \int_0^R \frac{6\,dx}{(x+1)(x+3)}$$

$$= \lim_{R\to\infty} 3\ln|x+1| - 3\ln|x+3| \Big|_0^R$$

$$= \lim_{R\to\infty} 3\left[(\ln|R+1| - \ln|R+3|) - (0 - \ln 3)\right]$$

$$= \lim_{R\to\infty} 3\left(\ln\left|\frac{R+1}{R+3}\right| + \ln 3\right) = 3\ln(1) + 3\ln 3$$

$$= 3\ln 3$$

3. Determine the limit of the sequence or show that the sequence diverges:

[7 points] (a) $a_n = \frac{n!}{3^n}$ (b) $a_n = n\,\sin(\frac{\pi}{2n})$ [13 points]

(a) $a_n = \frac{1}{3}\frac{2}{3}\left(\frac{3}{3}\frac{4}{3}\cdots\frac{n-1}{3}\right)\frac{n}{3} > \frac{2}{27}n$

because $\frac{3}{3}\frac{4}{3}\cdots\frac{n-1}{3} > 1$ and $\lim\limits_{n\to\infty}\frac{2}{27}n = \infty$

so $\{a_n\}$ diverges.

(b) $\lim\limits_{x\to\infty} x\sin\left(\frac{\pi}{2n}\right) = \lim\limits_{x\to\infty}\frac{\sin(\frac{\pi}{2}x^{-1})}{x^{-1}}$

$\underset{H}{=} \lim\limits_{x\to\infty}\frac{\cos(\frac{\pi}{2}x^{-1})(-\frac{\pi}{2}x^{-2})}{-x^{-2}}$

$= \lim\limits_{x\to\infty}\cos\left(\frac{\pi}{2}x^{-1}\right)\left(\frac{\pi}{2}\right) = \cos(0)\frac{\pi}{2} = \frac{\pi}{2}$

$\lim\limits_{n\to\infty} n\sin\left(\frac{\pi}{2n}\right) = \frac{\pi}{2}$

4. (a) Find the partial fraction decomposition of

[8 points]

$$\frac{2}{x^2+3x+2}$$

(b) Calculate the partial sums S_3 and S_4 of the series

[6 points]

$$\sum_{n=1}^{\infty} \frac{2}{n^2+3n+2}$$

(Hint: Use part (a).) (c) Find the sum S of the series of part (b). [6 points]

(a) $$\frac{2}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$$2 = (A+B)x + (2A+B) \quad B=-A \quad A=2 \quad B=-2$$

$$\frac{2}{x^2+3x+2} = \frac{2}{x+1} - \frac{2}{x+2}$$

(b) $$S_3 = \left(1-\frac{2}{3}\right)+\left(\frac{2}{3}-2\right)+\left(2-\frac{2}{5}\right) = 1-\frac{2}{5} = \frac{3}{5}$$

$$S_4 = \left(1-\frac{2}{3}\right)+\left(\frac{2}{3}-2\right)+\left(2-\frac{2}{5}\right)+\left(\frac{2}{5}-\frac{1}{3}\right) = \cancel{2}/3$$

(c) $$S_n = 1-\frac{2}{n+2}, \quad S = \lim_{n\to\infty} 1-\frac{2}{n+2} = 1$$

5. Evaluate the improper integrals

[7 points] (a) $\int_2^3 \frac{dx}{\sqrt{x-2}}$ (b) $\int_1^\infty \frac{dx}{\sqrt{x}e^{\sqrt{x}}}$ [13 points]

(a) $\int_2^3 \frac{dx}{(x-2)^{1/2}} = \lim_{R\to 2+} \int_R^3 (x-2)^{-1/2}dx$

$= \lim_{R\to 2+} 2(x-2)^{1/2}\Big|_R^3 = \lim_{R\to 2+} 2 - 2(R-2)^{1/2} = 2$

(b) $\int x^{-1/2} e^{-x^{1/2}} dx = \int e^{-u} 2du = -2e^{-u} = -2e^{-x^{1/2}}$

$u = x^{1/2}$ $du = \frac{1}{2}x^{-1/2}dx$ $x^{-1/2}dx = 2du$

$\int_1^\infty \frac{dx}{\sqrt{x}\, e^{\sqrt{x}}} = \lim_{R\to\infty} \int_1^R x^{-1/2} e^{-x^{1/2}} dx$

$= \lim_{R\to\infty} -2e^{-x^{1/2}}\Big|_1^R = \lim_{R\to\infty} -\frac{2}{e^{\sqrt{R}}} + \frac{2}{e} = \frac{2}{e}$