1. Differentiate

$$(a) \quad f(x) = 3^{\tan^{-1} x} \qquad\qquad (b) \quad f(x) = (x+1)^{(\ln x)^2}$$

[10 points] (a) $f(x) = \left(e^{\ln 3}\right)^{\tan^{-1} x} = e^{(\ln 3)\tan^{-1} x}$

$$f'(x) = e^{(\ln 3)\tan^{-1} x} (\ln 3) \frac{1}{1+x^2} \left(= \frac{\ln 3}{1+x^2} 3^{\tan^{-1} x}\right)$$

[10 points] (b) $y = (x+1)^{(\ln x)^2}$

$$\ln y = (\ln x)^2 \ln(x+1)$$

$$\frac{y'}{y} = 2(\ln x)\left(\frac{1}{x}\right)\ln(x+1) + (\ln x)^2 \frac{1}{x+1}$$

$$f'(x) = \left(\frac{2 \ln x \ln(x+1)}{x} + \frac{(\ln x)^2}{x+1}\right)(x+1)^{(\ln x)^2}$$

2. Let $f(x) = e^{e^x}$. (a) Show that $0 < f''(x) \le 2e$ for all x in $[-4, 0]$. (Hint: What does $f'''(x)$ tell you?) (b) Use the Error Bound for the Midpoint Rule to determine how large N must be to approximate

$$\int_{-4}^{0} e^{e^x}\,dx$$

by M_N to within 10^{-5}. (Leave your answer in the form $N =$ an expression that could be computed with a calculator.)

[13 points] (a) $f'(x) = e^{e^x}(e^x)$ $f''(x) = e^{e^x}(e^x)^2 + e^{e^x}(e^x)$

$0 < f''(x) \le f''(0) = e + e = 2e$ because $f'''(x) > 0$ so $f''(x)$ is increasing so $f''(x) \le f''(0)$ for x in $[-4, 0]$.

[7 points] (b) Error Bound $= \dfrac{(2e)(4)^3}{24N^2} \le 10^{-5}$

$$N^2 \ge \frac{(2e)(4)^3 10^5}{24}$$

$$N \ge \sqrt{\frac{(2e)(4)^3 10^5}{24}}$$

3. Calculate the limits.

$$(a) \quad \lim_{x\to 0+} (\cos x)^{1/x^2} \qquad\qquad (b) \quad \lim_{x\to\infty} xe^{1/x} - x$$

[10 points] (a) $y = (\cos x)^{1/x^2}$ $\quad \ln y = \dfrac{\ln(\cos x)}{x^2}$

$$\lim_{x\to 0+} \frac{\ln(\cos x)}{x^2} \underset{H}{=} \lim_{x\to 0+} \frac{\frac{-\sin x}{\cos x}}{2x}$$

$$= \lim_{x\to 0+} \frac{-\tan x}{2x} \underset{H}{=} \lim_{x\to 0+} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

so $\displaystyle\lim_{x\to 0+} (\cos x)^{1/x^2} = e^{-1/2}$

[10 points] (b) $\displaystyle\lim_{x\to\infty} x(e^{1/x} - 1)$

$$= \lim_{x\to\infty} \frac{e^{x^{-1}} - 1}{x^{-1}} \underset{H}{=} \lim_{x\to\infty} \frac{e^{x^{-1}}(-x^{-2})}{-x^{-2}}$$

$$= \lim_{x\to\infty} e^{1/x} = e^0 = 1$$

4. Evaluate the integrals

$$(a) \quad \int x \log_3(x^2+1)\,dx \qquad\qquad (b) \quad \int (\sin x)(\sinh x)\,dx$$

[7 points] (a) $\int x \log_3(x^2+1)\,dx = \int x \frac{\ln(x^2+1)}{\ln 3}\,dx$

(let $w = x^2+1$ so $dw = 2x\,dx$, $x\,dx = \frac{1}{2}dw$)

$= \frac{1}{2\ln 3}\int \ln w\,dw = \frac{1}{2\ln 3}\, w(\ln w - 1) + C$

$= \frac{1}{2\ln 3}(x^2+1)(\ln(x^2+1)-1) + C$

[13 points] (b) ($u = \sin x$ $du = \cos x\,dx$ $dv = \sinh x\,dx$
$v = \cosh x$)

$= (\sin x)(\cosh x) - \int (\cosh x)(\cos x)\,dx$

($u = \cos x$ $du = -\sin x\,dx$ $dv = \cosh x\,dx$ $v = \sinh x$)

$= (\sin x)(\cosh x) - \left((\cos x)(\sinh x) + \int (\sinh x)(\sin x)\,dx\right)$

$2\int (\sin x)(\sinh x)\,dx = (\sin x)(\cosh x) - (\cos x)(\sinh x)$
$+C$

$\int (\sin x)(\sinh x)\,dx = \frac{1}{2}\left((\sin x)(\cosh x) - (\cos x)(\sinh x)\right)$
$+C$

5. Calculate the limit

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \sin^{-1}(t^2)\, dt$$

$$\lim_{x \to 0} \frac{\int_0^x \sin^{-1}(t^2)\, dt}{x^3} \underset{H}{=} \lim_{x \to 0} \frac{\sin^{-1}(x^2)}{3x^2}$$

$$\underset{H}{=} \lim_{x \to 0} \frac{\frac{2x}{\sqrt{1-(x^2)^2}}}{6x} = \lim_{x \to 0} \frac{1}{3} \frac{1}{\sqrt{1-x^4}} = \frac{1}{3}$$