1. Determine whether the series is convergent or divergent.

[10 points] (a) $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5 - 2n^2}}$ (b) $\sum_{n=2}^{\infty} \frac{1}{n \log_2 n}$ [10 points]

(a)
$$\lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^5-2n^2}}}{\frac{n}{\sqrt{n^5}}} = \lim_{n\to\infty} \sqrt{\frac{n^5}{n^5-2n^2}} = \lim_{n\to\infty} \sqrt{\frac{1}{1-2/n^3}} = 1$$

$\frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}}$, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series,

so convergent by limit comparison test.

(b)
$$\sum_{n=2}^{\infty} \frac{1}{n \log_2 n} = \sum_{n=2}^{\infty} \frac{1}{n \frac{\ln n}{\ln 2}} = \ln 2 \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$$\int \frac{dx}{x \ln x} = \int u^{-1} du = \ln u = \ln(\ln x)$$
$$u = \ln x \quad du = x^{-1} dx$$

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{R\to\infty} \ln(\ln x)\Big|_2^R$$
$$= \lim_{R\to\infty} \ln(\ln R) - \ln(\ln 2) = \infty$$

so diverges by the integral test

2. Evaluate the integrals.

[10 points] (a) $\int \frac{x^2 - x}{(x+1)(x^2+1)}\,dx$ (b) $\int \cos(\ln x)\,dx$ [10 points]

(a) $\frac{x^2 - x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2+1} = \frac{Ax^2 + A + Bx^2 + Cx + Bx + C}{(x+1)(x^2+1)}$

$A + B = 1 \quad B + C = -1 \quad A + C = 0 \qquad A = 1 \quad B = 0 \quad C = -1$

$\int \frac{x^2 - x}{(x+1)(x^2+1)}\,dx = \int \frac{1}{x+1} - \frac{1}{x^2+1}\,dx = \ln|x+1| - \tan^{-1}x + C$

(b) $u = \cos(\ln x) \quad dv = dx \quad v = x$

$du = -\sin(\ln x)\frac{1}{x}$

$\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$

$u = \sin(\ln x) \quad dv = dx \quad v = x$

$du = \cos(\ln x)\,dx$

$= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$

$2\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x)$

$\int \cos(\ln x)\,dx = \frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$

3. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n2^n}(x-1)^n$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}}{(n+1)2^{n+1}}}{\frac{(-3)^n}{n2^n}} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{3^n}$$

$$= \lim_{n\to\infty} \frac{3}{2}\left(\frac{n}{n+1}\right) = \frac{3}{2} = \ell \text{ so } R = \frac{2}{3} \text{ and}$$

converges if $|x-1| < 2/3$

$x-1 = 2/3$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n2^n}\left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

converges by Leibnic alternating series test

$x-1 = -2/3$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n2^n}\left(-\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$

diverges (harmonic series)

interval of convergence $(1/3, 5/3]$

4. Calculate the limits.

[10 points] (a) $\lim_{x\to 0} \frac{\sin^{-1} x - x}{x^3}$ (b) $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$ [10 points]

(a) $$\lim_{x\to 0} \frac{\sin^{-1} x - x}{x^3} \underset{H}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}} - 1}{3x^2}$$

$$= \lim_{x\to 0} \frac{(1-x^2)^{-1/2} - 1}{3x^2} \underset{H}{=} \lim_{x\to 0} \frac{-\frac{1}{2}(1-x^2)^{-3/2}(-2x)}{6x}$$

$$= \lim_{x\to 0} \frac{(1-x^2)^{-3/2}}{6} = \frac{1}{6}$$

(b) $$y = \left(\frac{x}{x+1}\right)^x \quad \lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(\frac{x}{x+1}\right)$$

$$= \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{x^{-1}} \underset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{-x^{-2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x(x+1)}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{x^2}{x^2+x} = -1$$

$$\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = e^{-1}$$

5. Let $f(x) = \ln(3+x)$. (a) Find the Maclaurin polynomial $T_4(x)$ for $f(x)$. [10 points] (b) Find the radius of convergence R for the Maclaurin series $T(x)$ for $f(x)$. [10 points]

(a) $f(x) = \ln(3+x)$ $\quad f(0) = \ln 3$

$f'(x) = \frac{1}{3+x} = (3+x)^{-1}$ $\quad f'(0) = 1/3$

$f''(x) = -(3+x)^{-2}$ $\quad f''(0) = -1/9$

$f'''(x) = 2(3+x)^{-3}$ $\quad f'''(0) = 2/27$

$f^{(4)}(x) = -(2)(3)(3+x)^{-4}$ $\quad f^{(4)}(0) = -6/81$

$$T_4(x) = \ln 3 + \tfrac{1}{3}x - \frac{1/9}{2!}x^2 + \frac{2/27}{3!}x^3 - \frac{6/81}{4!}x^4$$

(b) $f^{(n)}(x) = (-1)^{n+1}(n-1)!\,(3+x)^{-n}$

$f^{(n)}(0) = (-1)^{n+1}(n-1)!\,3^{-n}$

$$T(x) = \ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)!\,3^{-n}}{n!} x^n$$

$$= \ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n3^n} x^n$$

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)3^{n+1}}}{\frac{1}{n3^n}} = \lim_{n\to\infty} \frac{n3^n}{n+1\,3^{n+1}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\frac{1}{3} = \frac{1}{3}$$

so $R = 3$

6. Find the maxima of the following functions on $[1, \infty)$. (You do not have to use a derivative test to show that it is the maximum.)

[10 points] (a) $f(x) = \frac{\ln x}{x^3}$ (b) $f(x) = xe^{-x^2/8}$ [10 points]

(a) $f'(x) = \dfrac{x^{-1}x^3 - (\ln x)3x^2}{x^6} = \dfrac{1 - 3\ln x}{x^4} = 0$

$3 \ln x = 1 \quad \ln x = 1/3 \quad x = e^{1/3}$

(b) $f'(x) = e^{-x^2/8} - x e^{-x^2/8}(x/4)$

$= e^{-x^2/8}\left(1 - \frac{x^2}{4}\right) = 0$

$\frac{x^2}{4} = 1 \quad x = \pm 2$ so $x = 2$

7. Determine whether the series converges absolutely, conditionally or not at all.

[10 points] (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{\cosh n}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\,\ln n}$ [10 points]

(a) $\left|\dfrac{\cos n}{\cosh n}\right| = \dfrac{2|\cos n|}{e^n + e^{-n}} < \dfrac{2}{e^n}$

$$\int_1^{\infty} e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_1^R = \lim_{R\to\infty} -\frac{1}{e^R} + \frac{1}{e} = \frac{1}{e}$$

so absolutely convergent by the comparison and integral tests

(b) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} \underset{H}{=} \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0$

so $\ln n < n^{1/2}$ for n large $\dfrac{1}{\ln n} > \dfrac{1}{n^{1/2}} = \dfrac{1}{\sqrt{n}}$

so $\dfrac{1}{\sqrt{n}\ln n} > \dfrac{1}{\sqrt{n}\sqrt{n}} = \dfrac{1}{n}$ and thus not absolutely convergent by the comparison test, but conditionally convergent by Leibniz alternating series test

8. Given that

$$\sum_{n=2}^{\infty}\left(\frac{a}{2}\right)^n = 2$$

determine the value of a.

$$\sum_{n=2}^{\infty}\left(\frac{a}{2}\right)^n = \left(\frac{a}{2}\right)^2 \frac{1}{1-\frac{a}{2}} = 2$$

$$\left(\frac{a}{2}\right)^2 = 2-a \qquad \frac{a^2}{4} = 2-a$$

$$a^2+4a-8=0 \qquad a = \frac{-4 \pm \sqrt{16+32}}{2}$$

$$= \frac{-4 \pm 4\sqrt{3}}{2} = -2 \pm 2\sqrt{3}$$

$a = -2+2\sqrt{3}$ because $|-2-2\sqrt{3}| > 2$

so $\displaystyle\sum_{n=2}^{\infty}\left(\frac{-2-2\sqrt{3}}{2}\right)^n$ diverges

9. Determine whether the improper integral converges or diverges. Do not attempt to evaluate the integrals.

$$(a) \quad \int_0^1 \frac{\ln(x+1)}{\sqrt{x}}\,dx \qquad\qquad (b) \quad \int_1^\infty \frac{dx}{x^{1/3}+x^{2/3}}$$

(a) $\ln(x+1) < 1$ for $0 \le x \le 1$ so

$\dfrac{\ln(x+1)}{\sqrt{x}} < \dfrac{1}{\sqrt{x}}$ and $\displaystyle\int_0^1 \frac{dx}{x^{1/2}}$

converges, so converges

(b) $x^{1/3} + x^{2/3} < x$ for x large

$\left(\lim\limits_{x\to\infty} \dfrac{x^{1/3}+x^{2/3}}{x} = 0\right)$ so $\dfrac{1}{x^{1/3}+x^{2/3}} > \dfrac{1}{x}$

and $\displaystyle\int_1^\infty \frac{dx}{x}$ diverges, so diverges

10. Calculate the area of the region of the plane bounded by the curves $y = xe^{x^2}$ and $y = x/e^x$.

$xe^{x^2} = \frac{x}{e^x} = xe^{-x}$ if $x = 0$ or $e^{x^2} = e^{-x}$,

That is, $x^2 = -x$, so $x = -1$. On $[-1, 0]$,

$xe^{x^2} \geq xe^{-x}$ (for instance $-\frac{1}{2}e^{1/4} > -\frac{1}{2}e^{1/2}$) so

$$\text{Area} = \int_{-1}^{0} xe^{x^2} - xe^{-x}\,dx = \int_{-1}^{0} xe^{x^2}dx - \int_{-1}^{0} xe^{-x}dx$$

$$\int xe^{x^2}dx = \int e^u \tfrac{1}{2}du = \tfrac{1}{2}e^u = \tfrac{1}{2}e^{x^2}$$

$u = x^2 \quad du = 2x\,dx$

$\quad x\,dx = \frac{1}{2}du$

$$\int_{-1}^{0} xe^{x^2}dx = \tfrac{1}{2}e^{x^2}\Big|_{-1}^{0} = \tfrac{1}{2} - \tfrac{1}{2}e$$

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$\int_{-1}^{0} xe^{-x}dx = -xe^{-x} - e^{-x}\Big|_{-1}^{0}$$

$$= (0 - 1) - (e - e) = -1$$

$$\text{Area} = \left(\tfrac{1}{2} - \tfrac{1}{2}e\right) - (-1) = \tfrac{3}{2} - \tfrac{e}{2}$$