1. Given the function

$$f(x) = x^4 - 2x^3 - 12x^2 + 5$$

determine where the function is concave upward and concave downward, and find the points of inflection, if there are any.

$$f'(x) = 4x^3 - 6x^2 - 24x$$

$$f''(x) = 12x^2 - 12x - 24$$

$$= 12(x^2 - x - 2)$$

$$= 12(x-2)(x+1)$$

$f''(x) = 0$ if $x = -1, 2$

	$x-2$	$x+1$	$f''(x)$	
$(-\infty, -1)$	neg	neg	pos	concave up
$(-1, 2)$	neg	pos	neg	concave down
$(2, \infty)$	pos	pos	pos	concave up

$-1, 2$ are points of inflection.

2. Consider the function

$$f(x) = \frac{\sqrt{2x^2 - 1}}{x}.$$

(a) Determine the domain of $f(x)$.

(b) Find the horizontal asymptotes of $f(x)$, if there are any.

(a) $2x^2 - 1 \geq 0 \qquad x^2 \geq \frac{1}{2} \qquad x \geq \sqrt{\frac{1}{2}}$ and $x \leq -\sqrt{\frac{1}{2}}$

(b) $$\lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{x} = \lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{\sqrt{x^2}}$$

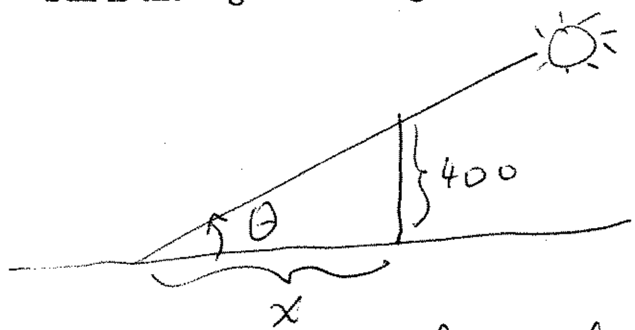
$$= \lim_{x \to \infty} \sqrt{\frac{2x^2-1}{x^2}} = \lim_{x \to \infty} \sqrt{2 - \frac{1}{x^2}}$$

$$= \sqrt{2 - \lim_{x \to \infty} \frac{1}{x^2}} = \sqrt{2}$$

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2-1}}{x} = \lim_{x \to \infty} \frac{\sqrt{2(-x)^2-1}}{-x}$$

$$= -\lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{x} = -\sqrt{2}$$

3. As the Sun sets, the angle of elevation of the Sun above the horizon is decreasing at the rate of $\frac{1}{4}$ radian/hr. How fast is the shadow cast by a 400-foot-tall building increasing when the angle of elevation of the Sun is $\frac{\pi}{4}$ radians. Note: the angle of elevation of the Sun is the angle from the ground - assumed flat - up to the Sun.



θ = angle of elevation $\qquad \frac{d\theta}{dt} = -\frac{1}{4}$

x = length of shadow

$$\tan\theta = \frac{400}{x} = 400x^{-1}$$

$$\sec^2\theta \frac{d\theta}{dt} = -400x^{-2}\frac{dx}{dt}$$

$$\sec^2(\pi/4) = \frac{1}{\cos^2(\pi/4)} = \left(\frac{1}{\sqrt{2}/2}\right)^2 = (\sqrt{2})^2 = 2$$

$\tan(\pi/4) = 1$ so $x = 400$

$$2\left(-\frac{1}{4}\right) = -400(400)^{-2}\frac{dx}{dt}$$

$$\frac{dx}{dt} = 200 \text{ ft/hr.}$$

4. Find all the local maxima and local minima of the function

$$f(x) = x^{2/3}(1-x)^2.$$

$$f'(x) = \frac{2}{3}x^{-1/3}(1-x)^2 + x^{2/3}\,2(1-x)(-1)$$

$$= 2x^{-1/3}\left(\frac{1}{3}(1-x)^2 - x(1-x)\right)$$

$$= 2x^{-1/3}\left(\frac{1}{3} - \frac{2}{3}x + \frac{1}{3}x^2 - x + x^2\right)$$

$$= 2x^{-1/3}\left(\frac{4}{3}x^2 - \frac{5}{3}x + \frac{1}{3}\right)$$

$$= \frac{2}{3}x^{-1/3}(4x^2 - 5x + 1)$$

$$= \frac{2}{3}x^{-1/3}(4x-1)(x-1)$$

Critical points $x = 0, \frac{1}{4}, 1$

	$x^{-1/3}$	$4x-1$	$x-1$	$f'(x)$
$(-\infty, 0)$	neg	neg	neg	neg
$(0, \frac{1}{4})$	pos	neg	neg	pos
$(\frac{1}{4}, 1)$	pos	pos	neg	neg
$(1, \infty)$	pos	pos	pos	pos

0, 1 are local min; $\frac{1}{4}$ local max

5. Suppose that a function $f(x)$ has second derivative $f''(x)$ for all real numbers x.

(a) Prove that there exists a point c in $(0,1)$ such that

$$f(1) - f(0) = f'(c).$$

(b) Prove that if $|f''(x)| \leq 1$ for all x in $(0,1)$, then

$$|f(1) - f(0) - f'(0)| < 1.$$

(a) By MVT

$$f(1) - f(0) = f'(c)(1-0) = f'(c)$$

(b) By part (a)

$$f(1) - f(0) - f'(0) = f'(c) - f'(0)$$

By MVT, exists d in $(0,c)$ with

$$f'(c) - f'(0) = f''(d)(c-0)$$

substituting

$$|f(1) - f(0) - f'(0)|$$

$$= |f'(c) - f'(0)| = |f''(d)||c-0| < 1$$

since $|f''(d)| \leq 1$ and $|c-0| < 1$.